A Note on the Nonholonomic Self-Adjoints in Vn

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감 사 의 글

그동안 많은 지도와 격려를 해주신 현진오교수님께 무한 한 감사를 드리며, 지도와 편달을 아끼지 않으신 수학과 여러 교수님과 동료들에게 감사를 드립니다.

그리고 항상 사랑으로 보살펴주시는 부모님과 가족들에게도 감사를 드립니다.

1982년 6월 일

강 태 철

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국　문　초　록

V_n 공간에서의 NONHOLONOMIC SELF-ADJOINT에 관한 소고

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이 논문의 주요한 목적은 HOLONOMIC과 NONHOLONOMIC COMPONENT 사이의 관계를 연구하고, 이 구조에 대한 몇가지 특수한 성질을 증명하였다.

1. INTRODUCTION

Let V_n be a n-dimensional Riemannian space referred to a real coordinate system X^ν and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

(1.1) $\quad h \overset{\text{def}}{=\!=} \text{Det}\,((h_{\lambda\mu})) \neq 0.$

According to (1.1), there is a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

(1.2) $\quad h_{\lambda\mu} h^{\lambda\nu} \overset{\text{def}}{=\!=} \delta^\nu_\mu$

Let $\underset{i}{e}{}^\nu$, $(i = 1, 2, \cdots\cdots, n)$, be a set of n linearly independent vectors.

Then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$, $(i = 1, 2, \cdots\cdots, n)$, satisfying

(1.3) a $\quad \underset{i}{e}{}^\nu \overset{i}{e}_\lambda = \delta^\nu_\lambda$ (*)

(1.3) b $\quad \underset{j}{e}{}^\lambda \overset{i}{e}_\lambda = \delta^i_j$

DEFINITION 1. 1) With the vectors $\underset{i}{e}{}^\nu$ and $\overset{i}{e}_\lambda$ a nonholonomic frame of V_n is defined in the following way ; If $T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}$ are holonomic

(*) ———

Throughout the present paper, Greek indices are used for the holonomic components of a tensor, while Roman indices are used for the nonholonomic components of a tensor.
Both indices take the values 1, 2, ⋯⋯, n, and follow the summation convention.

components of a tensor, then its nonholonomic components are defined by

$$(1.4)\ a \qquad T^{i\ \cdots\cdots}_{j\ \cdots\cdots} \overset{\text{def}}{=\!=} T^{\nu\ \cdots\cdots}_{\lambda\ \cdots\cdots}\ \overset{i}{e}_{\nu}\ \underset{j}{e}^{\lambda}\ \cdots\cdots.$$

An easy inspection of (1.3)a and (1.4)a show that

$$(1.4)\ b \qquad T^{\nu\ \cdots\cdots}_{\lambda\ \cdots\cdots} = T^{i\ \cdots\cdots}_{j\ \cdots\cdots}\ \underset{i}{e}^{\nu}\ \overset{j}{e}_{\lambda}\ \cdots\cdots.$$

2. PRELIMINARY RESULTS

In the present section, for our further discussions, results obtained in our previous paper will be introduced without proof.

THEOREM 2.1) The product of two nonholonomic components of $h_{\lambda\mu}$ and $h^{\lambda\nu}$ is kronecker delta.

$$(2.1) \qquad h_{ij}\, h^{ik} = \delta_j^{\,k}$$

THEOREM 2.2) We have

$$(2.2) \qquad \underset{i}{e}^{\nu} = \overset{j}{e}_{\lambda}\, h_{ij} h^{\lambda\nu}, \qquad \overset{j}{e}_{\lambda} = \underset{i}{e}^{\nu}\, h^{ij}\, h_{\lambda\nu}\,.$$

The nonholonomic frame in V_n constructed by the unit vectors $\underset{i}{e}^{\nu}$ tangent to the n congruences of an orthogonal ennuple, will be termed an orthogonal nonholonomic frame of V_n.

THEOREM 2.3) We have

$$(2.3)\,a \qquad h_{ij} = \delta_{ij}\,, \qquad h^{ij} = \delta^{ij}.$$

$$(2.3)\,b \qquad \underset{i}{e}^{\nu} = \overset{i\nu}{e}\,, \qquad \overset{j}{e}_{\lambda} = \underset{j}{e}_{\lambda}\,.$$

3. MAIN THEOREMS

In this section, we will study some of the relationships between holonomic and nonholonomic components, and derive a useful representation of the nonholonomic components.
Our further discussions will be restricted to an orthogonal nonholonomic frames only.
First of all, we shall derive some special properties of this frame in the following theorem.

THEOREM 3.1) We have

$$(3.1) \qquad \underset{i}{e}^{\nu} = \underset{i}{e}_{\nu}, \qquad \overset{j}{e}_{\lambda} = \overset{j}{e}^{\lambda}.$$

Proof). By means of (2.3)b and $\underset{i}{e}^{\nu}$ are mutually orthogonal unit vectors, easily obtained the results.

THEOREM 3.2) The nonholonomic components of the covariant $h_{\lambda\mu}$ and contravariant tensor $h^{\lambda\mu}$ expressed in terms of $\underset{i}{e}^{\lambda}$, as follows ;

$$(3.2) \qquad h^{\lambda\mu} = \underset{i}{e}^{\lambda} h^{ij} \underset{j}{e}^{\mu} = \overset{i}{e}_{\lambda} h_{ij} \overset{j}{e}_{\mu}.$$

Proof). Using (1.4)b, (2.3)a and (3.1), easily obtained the results.

DEFINITION 3.3) A symmetric covariant tensor a whose determinant $a \overset{def}{=} \mathrm{Det}\,((a_{\lambda\mu})) \neq 0$

defined by

(3.3) $a^{\lambda\nu} \underline{\underline{\text{def}}} \frac{A_{\lambda\nu}}{a}$ is a symmetric contravariant tensor

satisfying $a_{\lambda\mu}\, a^{\lambda\nu} = \delta^{\nu}_{\mu}$,

where $A_{\lambda\nu}$ is the cofactor of $a_{\lambda\nu}$ in a.

THEOREM 3.4) The derivative of $\underset{i}{e}^{\lambda}$ is negative self-adjoint .

That is,

(3.4)a $\partial_{\kappa}(\overset{i}{e}_{\lambda})\, \underset{j}{e}^{\mu} = -\partial_{\kappa}(\underset{j}{e}^{\mu})\, \overset{i}{e}_{\lambda}$.

Proof). Take a coordinate system y^{i} for which we have at a point p of V_{n}.

(3.4)b $\frac{\partial y^{i}}{\partial x^{\lambda}} = \overset{i}{e}_{\lambda}$, $\frac{\partial x^{\nu}}{\partial y^{i}} = \underset{i}{e}^{\nu}$

$$\begin{aligned}\partial_{\kappa}(\overset{i}{e}_{\lambda})\, \underset{j}{e}^{\mu} &= -(\overset{i}{e}_{\lambda})^{2}\, \partial_{\kappa}(\underset{i}{e}^{\lambda})\, \underset{j}{e}^{\mu} \\ &= -\delta^{\mu}_{\lambda}\, (\overset{i}{e}_{\mu})\overset{i}{e}_{\lambda}\, \underset{j}{e}^{\mu}\ \ \partial_{\kappa}(\underset{i}{e}^{\lambda}) \\ &= -\delta^{i}_{j}\ \overset{i}{e}_{\lambda}\, \partial_{\kappa}(\underset{i}{e}^{\mu}) \\ &= -\ \overset{i}{e}_{\lambda}\, \partial_{\kappa}(\underset{j}{e}^{\mu}).\end{aligned}$$

THEOREM 3.5) The derivative of the tensor $a_{\lambda\mu}$ is negative self - adjoint.

Proof). By means of (3.3), we derive the

(3.5) $a^{\lambda\mu}\, \partial_{\kappa}(a_{\lambda\mu}) = -\, a_{\lambda\mu}\ \partial_{\kappa}(a^{\lambda\mu})$.

THEOREM 3.6) The derivative of the nonholonomic components of $a_{\lambda\mu}$ is negative self-adjoint.

Proof). Using (1.4)a, (1.4)b, (3.3), (3.4)a, (3.5),

$$
\begin{aligned}
& a^{ij}\, \partial_{\kappa}\, (a_{ij}) + a_{ij}\, \partial_{\kappa}\, (a^{ij}) \\
= \; & a^{ij}\, \partial_{\kappa}\, (a_{\lambda\mu}\, \underset{i}{e}^{\lambda}\, \underset{j}{e}^{\mu}) + a_{ij}\, \partial_{\kappa}\, (a^{\lambda\mu}\, \overset{i}{e}_{\lambda}\, \overset{j}{e}_{\mu}) \\
= \; & a^{ij}\, \partial_{\kappa}\, (a_{\lambda\mu})\, \underset{i}{e}^{\lambda}\, \underset{j}{e}^{\mu} + a^{\lambda w}\, \overset{i}{e}_{\lambda}\, \overset{j}{e}_{w}\, a_{\lambda\mu}\, \partial_{\kappa}\, (\underset{i}{e}^{\lambda})\, \underset{j}{e}^{\mu} \\
& \qquad + a^{\lambda w}\, \overset{i}{e}_{\lambda}\, \overset{j}{e}_{w}\, a_{\lambda\mu}\, \underset{i}{e}^{\lambda}\, \partial_{\kappa}\, (\underset{j}{e}^{\mu}) \\
& + a_{ij}\, \partial_{\kappa}\, (a^{\lambda\mu})\, \overset{i}{e}_{\lambda}\, \overset{j}{e}_{\mu} + a_{\lambda w}\, \underset{i}{e}^{\lambda}\, \underset{j}{e}^{w}\, a^{\lambda\mu}\, \partial_{\kappa}\, (\overset{i}{e}_{\lambda})\, \overset{j}{e}_{\mu} \\
& \qquad + a_{\lambda w}\, \underset{i}{e}^{\lambda}\, \underset{j}{e}^{w}\, a^{\lambda\mu}\, \overset{i}{e}_{\lambda}\, \partial_{\kappa}\, (\overset{j}{e}_{\mu}) \\
= \; & a^{\lambda\mu}\, \partial_{\kappa}\, (a_{\lambda\mu}) + \overset{i}{e}_{\lambda}\, \partial_{\kappa}\, (\underset{i}{e}^{\lambda}) + \overset{j}{e}_{\mu}\, \partial_{\kappa}\, (\underset{j}{e}^{\mu}) \\
& + a_{\lambda\mu}\, \partial_{\kappa}\, (a^{\lambda\mu}) + \underset{i}{e}^{\lambda}\, \partial_{\kappa}\, (\overset{i}{e}_{\lambda}) + \underset{j}{e}^{\mu}\, \partial_{\kappa}\, (\overset{j}{e}_{\mu}) \\
= \; & a^{\lambda\mu}\, \partial_{\kappa}\, (a_{\lambda\mu}) + a_{\lambda\mu}\, \partial_{\kappa}\, (a^{\lambda\mu}) .
\end{aligned}
$$

By the theorem (3.4)b, we have the result.

COROLLARY 3.7) The negative self-adjoint of the derivative of the tensor $a_{\lambda\mu}$ is equal to its nonholonomic components.

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ABSTRACT

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The purpose of the present paper is to study some of the relationships between holonomic and nonholonomic components, and so derive some special properties of this frame.