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Extreme Preservers of Matrix Rank Inequalities

濟州大學校　大學院

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Extreme Preservers of Matrix Rank Inequalities

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Extreme Preservers of Matrix Rank Inequalities

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⟨**Abstract**⟩

Extreme Preservers of Matrix Rank Inequalities

In this thesis, we construct the sets of real matrix pairs. These sets are naturally occurred at the extreme cases for the real rank inequalities relative to the sums and products of real matrices. These sets were constructed with real matrix pairs which are related with the ranks of the sums and products of two real matrices. For a given square matrix A of order n, we denote the real rank of A by $\rho(A)$. Then for $n \times n$ matrices A, B and C, we construct the following sets;

$\sum_1 = \{(A, B) \,|\, \rho(A + B) = \rho(A) + \rho(B)\}$;

$\sum_2 = \{(A, B) \,|\, \rho(A + B) = |\rho(A) - \rho(B)|\}$;

$\mathfrak{M}_3 = \{(A, B) \,|\, \rho(AB) = \min\{\rho(A), \rho(B)\}\}$;

$\mathfrak{M}_4 = \{(A, B) \,|\, \rho(AB) = \rho(A) + \rho(B) - n\}$;

$\mathfrak{M}_5 = \{(A, B, C) \,|\, \rho(AB) + \rho(BC) = \rho(ABC) + \rho(B)\}$.

For these sets of real matrix pairs, we consider the linear operators that preserve them. We characterize those linear operators as $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ by $T(X) = \alpha P X P^{-1}$ with appropriate invertible matrix P and scalar α. We also prove that these linear operators preserve above sets of matrix pairs.

1 Introduction and Preliminaries

During the last century, there are many research works on linear operators on matrices that leave certain properties or relations of subsets invariant. Such questions are usually called "Linear Preserver Problems". The earliest papers in our reference list are Frobenius(1897)and Kantor(1897). Since much effort has been devoted to this type of problem, there have been several excellent survey papers. For survey of these types of problems, we refer to the article of Song([8]) and the papers in [7]. The specified frame of problems is of interest both for matrices with entries from a field and for matrices with entries from an arbitrary semiring such as Boolean algebra, nonnegative integers, and fuzzy sets. There are some results on the inequalities for the rank function of matrices([1,2,3,5,6]). Beasley and his colleagues investigated the rank inequalities of matrices over semirings, and they characterized the equality cases for some rank inequalities in [2,3,5]. The investigation of linear preserver problems of extreme cases of the rank inequalities of matrices over fields was obtained in [2]. The structure of matrix varieties which arise as extremal cases in the inequalities is far from being understood over fields, as well as semirings. A usual way to generate elements of such a variety is to find a matrix pairs which belongs to it and to act on this set by various linear operators that preserve this variety. Song an his colleagues([3]) characterized the linear operators that preserve the extreme cases of column rank inequalities over semirings.

In this thesis, we characterize linear operators that preserve the sets of matrix pairs which satisfy extreme cases for the rank inequalities for the product of matrices over real fields.

The matrix I_n is the $n \times n$ identity matrix, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with exactly one nonzero entry, that being a 1 in the (i, j) entry.

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with entries in the reals $\mathbb{R}$. Let $\rho(A)$ denote the *real rank* of $A \in \mathcal{M}_n(\mathbb{R})$. If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ is a linear operator, we say that T is a $(U, V) - operator$ provided there exist nonsingular matrices U and V in $\mathcal{M}_n(\mathbb{R})$ such that either

(1) $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbb{R})$ or

(2) $T(X) = UX^tV$ for all $X \in \mathcal{M}_n(\mathbb{R})$,

where X^t denotes the transpose of X

Some classical inequalities concerning the rank of sums and products of A, B and C in $\mathcal{M}_n(\mathbb{R})$ are :

The *rank-sum inequalities*:

$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq \rho(A) + \rho(B)$$

Sylvester's laws:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

Frobenius inequality:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B).$$

In this paper we shall investigated linear operators which preserve pairs or triples of matrices which attain one of the extremes of the inequalities above.

Let

$\sum_1 = \{(A, B) \,|\, \rho(A + B) = \rho(A) + \rho(B)\}$;

$\sum_2 = \{(A, B) \,|\, \rho(A + B) = |\rho(A) - \rho(B)|\}$;

$\mathfrak{M}_3 = \{(A, B) \,|\, \rho(AB) = \min\{\rho(A), \rho(B)\}\}$;

$\mathfrak{M}_4 = \{(A, B) \,|\, \rho(AB) = \rho(A) + \rho(B) - n\}$;

$\mathfrak{M}_5 = \{(A, B, C) \,|\, \rho(AB) + \rho(BC) = \rho(ABC) + \rho(B)\}$.

It was shown in [1,2,3] that linear operators that preserve $\sum_1$ or $\sum_2$ are (U, V)-operators. Here we investigate linear operators that preserve $\mathfrak{M}_3$, $\mathfrak{M}_4$ or $\mathfrak{M}_5$. In order to characterize linear preservers for these extreme rank conditions some results on rank preservers are vital: The following lemma is Lemma from Beasley and Laffey [4].

Lemma 1.1. [4] *Let* $\mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ *be an invertible linear operator that preserves the set of rank-*n *matrices, or the set of rank-*1 *matrices. Then* T *is a* (U, V)*-operator.*

We use this Lemma 1.1 in order to show our main results.

2 Preservers of the set of matrix pairs in the upper extreme rank of matrix product.

In this section, we study the linear operator that preserve $\mathfrak{M}_3 = \{(A, B) \mid \rho(AB) = \min\{\rho(A), \rho(B)\}\}$, the set of matrix pairs in the upper extreme rank of two matrix product. We begin with an example that preserve $\mathfrak{M}_3$.

Example 2.1. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined by $T(X) = 2X = 2IXI^{-1}$. If

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and so $(A, B) \in \mathfrak{M}_3$ because $\rho(AB) = 1 = \min\{2, 1\} = \min\{\rho(A), \rho(B)\}$. Also, we have $(T(A), T(B)) \in \mathfrak{M}_3$ because $\rho(T(A)T(B)) = \rho(4AB) = 1 = \min\{\rho(T(A)), \rho(T(B))\}$. Therefore T preserves $\mathfrak{M}_3$ in $\mathcal{M}_2(\mathbb{R})$.

We need two Lemmas in order to obtain the linear preservers of $\mathfrak{M}_3$.

Lemma 2.2. *If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ preserves the set $\mathfrak{M}_3$ and T is invertible, then T preserves the set of rank-1 matrices.*

Proof. Suppose that T^{-1} does not preserve matrices of rank 1. Then there is a matrix A such that $\rho(A) = k$ and $\rho(T(A)) = 1$. Since similarity operators preserve $\mathfrak{M}_3$, we may assume without loss of generality that

$$A = \begin{bmatrix} A_1 \\ O \end{bmatrix},$$

where A_1 is a $k \times n$ matrix of rank k, and

$$T(A) = \begin{bmatrix} \boldsymbol{a}^t \\ O \end{bmatrix},$$

where $\boldsymbol{a}^t$ is a certain nonzero row of $T(A)$.

Now, if $\mathcal{H}$ is a space of matrices such that for each nonzero $H \in \mathcal{H}$, $HT(A) \neq O$, we must have that $\dim \mathcal{H} \leq n$. (The dimension of the complement of $\mathcal{H}$ is greater than or equal to $n(n-1)$ since all matrices with zero first column and arbitrary columns from 2^{nd} till n^{th} annihilate $T(A)$).

Let $\mathcal{K} = \{B = [B_1 \;\; O] \in \mathcal{M}_n(\mathbb{R}) \mid B_1 \text{ is } n \times k\}$. Then $\dim \mathcal{K} = kn$. If $B \in \mathcal{K}$ then $\rho(BA) = \rho(B) = \min\{(\rho(A), \rho(B)\}$ and so $(B, A) \in \mathfrak{M}_3$. It follows that $(T(B), T(A)) \in \mathfrak{M}_3$ so that $\rho(T(B)T(A)) = \min\{\rho(T(B)), \rho(T(A))\} = 1$. Thus for each $C \in T(\mathcal{K})$, we have $\rho(CT(A)) = 1$ or $CT(A) \neq O$. It follows from the above observation that $\dim T(\mathcal{K}) \leq n$. But T is invertible so that $dimT(\mathcal{K}) = nk$, a contradiction. Thus T^{-1}, and hence T preserves the set of rank-1 matrices. □

Lemma 2.3. *Let $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be defined by $T(X) = UXV$ for some invertible matrices U and V. Then T preserves the set $\mathfrak{M}_3$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ and for some nonzero real α.*

Proof. ($\Longleftarrow$) It is easy to check that the transformation $T(X) = \alpha PXP^{-1}$ preserves the set $\mathfrak{M}_3$.

($\Longrightarrow$) It is enough to consider transformations of the form $X \to XD$, where D is an arbitrary invertible matrix, instead of $T(X) = UXV$ since the similarity transformation preserves $\mathfrak{M}_3$ and $U^{-1}T(X)U = XVU = XD$. To prove the lemma we need to show that the matrix $D = [d_{ij}]$ is a scalar matrix.

Step 1. First, we show that $d_{ii} \neq 0$ for all $i = 1, \ldots, n$. For arbitrary i, we consider the matrices $A_1 = E_{i,i}$ and $B_1 = E_{i,j}$ for some $j \neq i$. Then $(A_1, B_1) \in \mathfrak{M}_3$ because $\rho(A_1B_1) = 1 = \rho(A_1) = \rho(B_1)$. Since D is invertible, we have that $\rho(A_1D) = 1$, $\rho(B_1D) = 1$ and $\rho(A_1DB_1D) = \rho(A_1DB_1)$. Hence $A_1DB_1 \neq 0$. On the other hand, $A_1D = d_{i1}E_{i,1} + \cdots + d_{in}E_{i,n}$. Thus $A_1DB_1 = d_{ii}E_{i,j}$. Therefore, $d_{ii} \neq 0$ for all $i = 1, \ldots, n$.

Step 2. Now we will show that $d_{ij} = 0$ for all i and j with $i \neq j$. Suppose that $d_{ij} \neq 0$ for some $i \neq j$. Consider the matrices $A_2 = E_{j,j} - \frac{d_{jj}}{d_{ij}}E_{j,i}$ and $B_2 = E_{j,i}$. Then $A_2B_2 = E_{j,i}$ and so $\rho(A_2) = \rho(B_2) = \rho(A_2B_2) = 1$. Hence $(A_2, B_2) \in \mathfrak{M}_3$. Therefore, $(A_2D, B_2D) \in \mathfrak{M}_3$. Since D is invertible, we have $\rho(A_2D) = 1$ and $\rho(B_2D) = 1$. Then $\rho(A_2DB_2D) = 1$ and hence $\rho(A_2DB_2D) = \rho(A_2DB_2) = 1$.

On the other hand,

$$\begin{aligned} A_2DB_2 &= \left(E_{j,j} - \frac{d_{jj}}{d_{ij}}E_{j,i}\right)DE_{j,i} \\ &= E_{j,j}DE_{j,i} - \frac{d_{jj}}{d_{ij}}E_{j,i}DE_{j,i} \\ &= d_{jj}E_{j,i} - \frac{d_{jj}}{d_{ij}}(d_{i1}Ej,1 + \ldots.. + d_{in}E_{j,n})E_{j,i} \\ &= d_{jj}E_{j,i} - \frac{d_{jj}}{d_{ij}}d_{ij}E_{j,j}E_{j,i} \\ &= d_{jj}E_{j,i} - d_{jj}E_{j,i} \\ &= 0. \end{aligned}$$

Thus $\rho(A_2DB_2) = 0$, a contradiction. Thus D is a diagonal matrix.

Step 3. It remains to check that $D = \text{diag}(d_{11}, \ldots, d_{nn})$ is indeed a scalar matrix. Assume that D is not scalar. Then there is an index i such that $d_{ii} \neq d_{i+1i+1}$. Consider the block-diagonal matrices

$$A_3 = \begin{bmatrix} I_{i-1} & O & O \\ O & L & O \\ O & O & L_{n-i-1} \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} I_{i-1} & O & O \\ O & M & O \\ O & O & L_{n-i-1} \end{bmatrix},$$

where L and M are 2×2 matrices of the following form:

$$L = \begin{bmatrix} d_{i+1i+1} & d_{ii} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

and I_k denotes the identity matrix of size k.

Thus we have $\rho(A_3) = n-1, \rho(B_3) = n-1$,

$\rho(A_3B_3) = n-1$ if and only if $d_{ii} \neq d_{i+1i+1}$, Thus $(A_3B_3) \in \mathfrak{M}_3$. On the other hand,

$$A_3DB_3 = \begin{bmatrix} d_{11} & & & & & & & \\ & \ddots & & & & & & \\ & & d_{i-1i-1} & & & & & \\ & & & 0 & 0 & & & \\ & & & 0 & 0 & & & \\ & & & & & d_{i+2i+2} & & \\ & & & & & & \ddots & \\ & & & & & & & d_{nn} \end{bmatrix}$$

It means that $\rho(A_3DB_3D) = n - 2$.

Since D is invertible thus $(T(A_3), T(B_3)) \notin \mathfrak{M}_3$.

Hence, $(A_3D, B_3D) \notin \mathfrak{M}_3$. This is a contradiction. to the assumption that T preserves $\mathfrak{M}_3$.

Therefore D is scalar matrix since similarity operator preserves rank and $T'(X) = XD$ implies $D = \alpha I$: Scalar matrix,

We have $T(X) = UT'(x)U^{-1} = UXDU^{-1} = UX(\alpha I)U^{-1} = \alpha UXU^{-1}$. Thus there exists an invertible matrix U such that $T(X) = \alpha UXU^{-1}$ □

We obtain the characterization of linear operators that preserve $\mathfrak{M}_3$.

Theorem 2.4. *If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ is invertible linear transformation then T preserves the set $\mathfrak{M}_3$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ and non-zero real α.*

Proof. $(\Longrightarrow)$ Assume that T is invertible and T preserves $\mathfrak{M}_3$. Then by Lemma 2.2, T preserves the set of rank-1 matrix. By lemma 1.1, T is a $(U, V)-operator$. If $T(X) = UXV$, by lemma 2.3, we have $T(X) = \alpha PXP^{-1}$. If $T(X) = UX^tV$, then consider three cell matrices $E_{i,j}, E_{j,k}, E_{i,k}$ with $k \neq i \neq j$. Let $T_1(X) = UXV$, $T_2(X) = X^t$ for all $X \in \mathcal{M}_n(\mathbb{R})$. Since $(E_{ij}, E_{jk}) \in \mathfrak{M}_3$, but $\rho(T_2(E_{ij}) \cdot T_2(E_{jk})) = \rho(E^t_{ij} \cdot E^t_{jk}) = \rho(E_{ji} \cdot E_{kj}) = \rho(0) = 0$. Hence T_2 does not preserve $\mathfrak{M}_3$. Hence $T = T_1 \circ T_2$ does not preserve $\mathfrak{M}_3$. Therefore $T(X) = \alpha PXP^{-1}$.

$(\Longleftarrow)$ It was proved in Lemma 2.3 □

Finally we remark that linear preservers of $\mathfrak{M}_3$ may be singular and non-trivial even over algebraically closed fields.

Example 2.5. Let $T : \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ be defined by $T(E_{1,1}) = E_{1,1}$, $T(E_{1,2}) = E_{1,2} + E_{2,1}$, and $T(E_{i,j}) = O$ for all $(i, j) \neq (1, 1)$ or $(1, 2)$. Let $A, B \in \mathcal{M}_3(\mathbb{R})$, with $\rho(A) = \rho(B) = \rho(AB) = 1$ such that

$$A = \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} c & d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $abcd \neq 0$. Then

$$AB = \begin{bmatrix} ac & ad & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A, B) \in \mathfrak{M}_3$ and

$$T(A)T(B) = \begin{bmatrix} a & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ac+bd & ad & 0 \\ bc & bd & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $\rho(T(A)) = \rho(T(B)) = \rho(T(A)T(B)) = 2$, that is, $(T(A), T(B)) \in \mathfrak{M}_3$. But

$$T(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3.$$

Thus T is not invertible.

Thus in this section, we obtained the characterizations of the linear operator that preserve the set of matrix pairs in the upper extreme rank of matrix product.

3 Preservers of the set of matrix pairs in the lower extreme rank of matrix product.

In this section, we study the linear operators that preserve $\mathfrak{M}_4 = \{(A,B)\,|\,\rho(AB) = \rho(A)+\rho(B)-n\}$, the set of matrix pairs in the lower extreme rank of two matrix product. We begin with an example that preserve $\mathfrak{M}_4$.

Example 3.1. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined by $T(X) = 3X = 3IXI^{-1}$. If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then $AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and so $(A,B) \in \mathfrak{M}_4$ because $\rho(AB) = 1 = \rho(A) + \rho(B) - 2$. Also, we have $(T(A), T(B)) \in \mathfrak{M}_4$ because $\rho(T(A)T(B)) = \rho(9AB) = 1 = \rho(T(A)) + \rho(T(B)) - 2$. Therefore T preserves $\mathfrak{M}_4$ in $\mathcal{M}_2(\mathbb{R})$.

We begin with two Lemmas in order to obtain the linear preservers of $\mathfrak{M}_4$.

Lemma 3.2. *If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ preserves the set $\mathfrak{M}_4$ then T preserves the set of rank-n matrices.*

Proof. Let $A = O$ and let B be any nonsingular matrix. Thus $r(B) = n$. Then, $\rho(A) = 0$ and $\rho(B) = n$ so that $\rho(AB) = \rho(A) + \rho(B) - n$. Therefore $(A,B) \in \mathfrak{M}_4$ and $(T(A), T(B)) \in \mathfrak{M}_4$ by assumption. Hence $\rho(T(A)T(B)) = \rho(T(A)) + \rho(T(B)) - n$. That is, $0 = 0 + \rho(T(B)) - n$. Therefore $\rho(T(B)) = n$. Hence T preserves rank-n matrices. □

Lemma 3.3. *Let $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be defined by $T(X) = UXV$ for some invertible matrices U and V. Then T preserves the set $\mathfrak{M}_4$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix P and non-zero real α.*

Proof. ($\Longleftarrow$) Let $T(X) = \alpha PXP^{-1}$. For any $(A,B) \in \mathfrak{M}_4$, $\rho(AB) = \rho(A) + \rho(B) - n$. Consider $T(A)T(B) = \alpha PAP^{-1} \cdot \alpha PBP^{-1} = \alpha^2 PABP^{-1}$. Therefore $\rho(T(A)T(B)) = \rho(\alpha^2 PABP^{-1}) = \rho(PABP^{-1}) = \rho(AB)$. Similarly $\rho(T(A)) =$

$\rho(A)$ and $\rho(T(B)) = \rho(B)$. Hence $\rho(T(A)T(B)) = \rho(AB) = \rho(A) + \rho(B) - n = \rho(T(A)) + \rho(T(B)) - n$. Therefore $(T(A), T(B)) \in \mathfrak{M}_4$.

($\Longrightarrow$) Similarity preserves rank of any matrix in $\mathcal{M}_n(\mathbb{R})$ Hence similarity preserves $\mathfrak{M}_4$. Since $T(X) = UXV$, we denote $T'(X) = U^{-1}T(X)U = XVU = XD$. Then T' preserves $\mathfrak{M}_4$.

Claim : D : scalar matrix.

1. Claim D : diagonal matrix.

For any $1 \le i \le n$, we denote $J_i = I - E_{ii}$. Let us take $A_i = E_{i,i}, B_i = J_i$. We denote

$$D_i = B_iD = \begin{bmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \vdots \\ \mathbf{d_{i-1}} \\ \mathbf{0} \\ \mathbf{d_{i+1}} \\ \vdots \\ \mathbf{d_n} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ & & \ddots & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}. \qquad (*)$$

Consider

$$A_iB_i = \begin{bmatrix} O & & O \\ & 1 & \\ O & & O \end{bmatrix} \begin{bmatrix} 1 & & & & & O \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ O & & & & & 1 \end{bmatrix} = \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 0 & \cdots & 0 & \cdots & 0 \\ & & 0 & & \\ & & 0 & & \end{bmatrix} = O$$

Then $\rho(A_iB_i) = 0 = \rho(A_i) + \rho(B_i) - n$. Therefore $(A_i, B_i) \in \mathfrak{M}_4$ and hence $(T'(A_i), T'(B_i)) \in \mathfrak{M}_4$, by assumption. Then $\rho(A_iDB_i) = 0$. Therefore $(T'(A_i), T'(B_i)) \in \mathfrak{M}_4$, by assumption. Hence $\rho(A_iD \cdot B_iD) = \rho(A_iD) + \rho(B_iD) - n = 0$. Therefore $A_iD \cdot B_iD = O$. Since $\mathbf{d_i}D_i$ is the i^{th} row of A_iDB_iD, we have $d_iD_i = 0$. Hence $\mathbf{d_i}$ is perpendicular to all columns of D_i. That can only happen if $d_{ij} = 0$ for all $j \neq i$. Since D is nonsingular we have that $d_{i,i} \neq 0$. That is D is a nonsingular diagonal matrix.

2. Claim: D is a scalar matrix.

$$A'_i = E_{ii} + E_{i,i+1} = \begin{bmatrix} & & & \\ & & & \\ & 1 & 1 & \\ & & & \\ & & & \end{bmatrix},$$

$$B'_i = E_{1,1} + \cdots + E_{i,i} + E_{i+2,i+2} + \cdots + E_{n,n} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{bmatrix}.$$

Then $A'_i B'_i = E_{i,i} - E_{i,i} = O$, $\rho(A'_i) + \rho(B'_i) = 1 + (n-1) = n$. So $\rho(A'_i \cdot B'_i) = \rho(A'_i) + \rho(B'_i) - n = 0$. Hence $(A'_i, B'_i) \in \mathfrak{M}_4$. By assumption, $(T'(A'_i), T'(B'_i)) \in \mathfrak{M}_4$ That is $(A'_i D, B'_i D) \in \mathfrak{M}_4$. Thus $\rho(A'_i D) = \rho(A'_i)$, $\rho(B'_i D) = \rho(B'_i)$ since D is nonsingular. And hence

$$\begin{aligned} \rho(A'_i D B'_i D) &= \rho(A'_i D) + \rho(B'_i D) - n \\ &= \rho(A'_i) + \rho(B'_i) - n \\ &= 1 + (n-1) - n \\ &= 0. \end{aligned}$$

Therefore $A'_i D B'_i D = 0$. On the other hand, one has $A'_i D B'_i = (E_{i,i} +$

$E_{i,i+1})(d_{11}E_{1,1}+\cdots+d_{nn}E_{n,n}) = (E_{1,1}+\cdots+E_{i,i}-E_{i+1,i}+E_{i+2,i+2}+\cdots+E_{n,n})$

$$= \begin{bmatrix} \\ 0 & \cdots & 1 & 1 & 0 & \cdots \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} d_{11} & & & & & \\ & \ddots & & & & \\ & & d_{ii} & & & \\ & & & d_{i+1,i+1} & & \\ & & & & \ddots & \\ & & & & & \ddots \\ & & & & & & d_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 0 & & & \\ & & -1 & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ 0 & \cdots & d_{ii}, d_{i+1,i+1} & 0 & 0 & \cdots & 0 \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & -1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

$= (d_{ii}E_{i,i}+d_{i+1i+1}E_{i,i+1})(E_{1,1}+\cdots+E_{i,i}-E_{i+1,i}+E_{i+2,i+2}+\cdots+E_{n,n})$

$$= \begin{bmatrix} \\ 0 & \cdots & 0 & d_{ii}-d_{i+1,i+1} & 0 & \cdots & 0 \\ \\ \\ \\ \end{bmatrix}$$

$= (d_{ii}-di+1i+1)E_{ii}$. Therefore $d_{ii}=d_{i+1i+1}$ for all $i=1......n-1$.

Thus D is a scalar matrix, say αI as we claimed.

Now $T'(X) = XD = X\alpha I = \alpha XI$. From $T(X) = UXV$, we have $U^{-1}T(X)U = XVU = XD = X(\alpha I)$. Now $T(X) = U(U^{-1}T(X)U)U^{-1} = U(XD)U^{-1} = U(X(\alpha I))U^{-1} = \alpha UXU^{-1}$. If we take $U = P$, then we have $T(X) = \alpha PXP^{-1}$ for some invertible matrix P and nonzero real α. □

Now, we obtain the characterization of linear operators that preserve $\mathfrak{M}_4$.

Theorem 3.4. *The bijective linear transformation $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ preserves the set $\mathfrak{M}_4$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ and non-zero real α.*

Proof. ($\Longleftarrow$) It is similar to the proof of lemma 3.3. That is, if $T(X) = \alpha PXP^{-1}$ for some invertible $P \in \mathcal{M}_n(\mathbb{R})$ then T preserves $\mathfrak{M}_4$.

($\Longrightarrow$)By Lemma 3.2, T preserves the set of rank-n matrices. By Lemma 1.1, T is a (U, V)-operator since we assume T is invertible. That is, T is a composition of operators of the form:

$T(X) = UXV$ for some nonsingular matrices U and V; or

$T(X) = X^t$ where X^t denotes the transpose of X.

If $T(X) = UXV$, then $U^{-1}T(X)U = XVU = XD$ for some nonsingular scalar matrix D.

Note that

$$J_j^t = (I - E_{i,i})^t = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} = J_j$$

for all $j = 1, \ldots\ldots, n$. Consider of pair of matrices (E_{ij}, J_j). We have $\rho(E_{ij} \cdot J_j) = \rho(0) = 0$, $\rho(E_{ij}) = 1$ and $\rho(J_j) = n-1$. Hence $(E_{ij}, J_j) \in \mathfrak{M}_4$. But $(E_{ij}^t, J_j^t) = (E_{ji}, J_j)$ satisfies that $\rho(E_{ji} \cdot J_j) = \rho(E_{ji}) = 1$, $\rho(E_{ji}) = 1$ and $\rho(J_j) = n-1$. Hence $(E_{ij}^t, J_j^t) \notin \mathfrak{M}_4$. This shows that $T(X) = X^t$ does not preserve $\mathfrak{M}_4$. Hence $T(X)$ has the form UXV only. By Lemma 3.3 $T(X) = \alpha PXP^{-1}$ for some invertible matrix P and nonzero real α. □

Thus in this section, we obtained the characterizations of the linear operator that preserve the set of matrix pairs in the lower extreme rank of two matrix product.

4 Preservers of the set of matrix triples in the extreme rank of matrix product.

In this section, we study the linear operators that preserve $\mathfrak{M}_5 = \{(A, B, C) \mid \rho(AB) + \rho(BC) = \rho(ABC) + \rho(B)\}$, the set of matrix triples in the extreme rank of three matrix product. We begin with an example that preserve $\mathfrak{M}_5$.

Example 4.1. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined by $T(X) = 2X = 2IXI^{-1}$. If

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad ABC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and so $\rho(AB) + \rho(BC) = \rho(ABC) + \rho(B)$. Thus $(A, B, C) \in \mathfrak{M}_5$.

Also, we have $(T(A), T(B), T(C)) \in \mathfrak{M}_5$. because $\rho(T(A)T(B)) + \rho(T(B)T(C)) = \rho(T(A)T(B)T(C)) + \rho(T(B))$. Therefore T preserves $\mathfrak{M}_5$ in $\mathcal{M}_2(\mathbb{R})$.

We begin with five Lemmas in order to obtain the linear preservers of $\mathfrak{M}_5$.

Lemma 4.2. *Let $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be a bijective linear transformation that maps $\mathfrak{M}_5$ into $\mathfrak{M}_5$. Then T preserves invertible matrices.*

Proof. Consider the triple $(A, B, C) \in \mathcal{M}_n(\mathbb{R})^3$ where $A = O$, B is arbitrary and C is invertible. Then $\rho(AB) = \rho(0) = 0$, $\rho(BC) = \rho(B)$ and $\rho(ABC) = \rho(0) = 0$. Hence we have $(A, B, C) \in \mathfrak{M}_5$. Then $(T(A), T(C)) \in \mathfrak{M}_5$ by assumption. That is $\rho(T(A)T(B)) + \rho(T(B)T(C)) = \rho(T(A)T(B)T(C)) + \rho(T(B))$.

However, $T(A) = O$ since $A = O$ and T is linear. Thus one has $\rho(T(B)T(C)) = \rho(T(B))$ for all matrices B. Since T is bijective, it follows that $T(C)$ is invertible. Indeed, $T(B)$ runs through all $\mathcal{M}_n(\mathbb{R})$ as for as B does. If $T(C)$ is singular, then there is a nonzero B such that $T(B)T(C) = O$ and $\rho(T(B)T(C)) \neq \rho(T(B))$, which is a contradiction. Thus $T(C)$ is invertible, that is, T preserves invertible matrices. □

Our next lemmas will show that preservers of $\mathfrak{M}_5$ are indeed invertible.

Lemma 4.3. *If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ is a linear transformation which preserves the set $\mathfrak{M}_5$ then there are no rank-n matrices in kernel T if T is not zero map.*

Proof. Suppose to the contrary T preserves $\mathfrak{M}_5$ and $T(A) = O$ for some A with $\rho(A) = n$. Then $\rho(AB) + \rho(BA) = \rho(ABA) + \rho(B)$ for any $B \in \mathcal{M}_n(\mathbb{R})$. Thus $(A, B, A) \in \mathfrak{M}_5$, and hence $(T(A), T(B), T(C)) \in \mathfrak{M}_5$. Thus $(O, T(B), O) \in \mathfrak{M}_5$. Hence $\rho(O \cdot T(B)) + \rho(T(B) \cdot O) = \rho(O \cdot T(B) \cdot O) + \rho(T(B))$ and $\rho(T(B)) = 0$. Thus $T(B) = O$. This implies $T = 0$, the zero map, a contradiction. Thus there are no rank-n matrices in kernel of T. □

Lemma 4.4. *If A is an $n \times n$ matrix of rank-k then for some positive integers k_1 and k_2 such that $k_1 + k_2 = k$, A is similar to a matrix of the form*

$$\begin{bmatrix} X & O \\ O_{k-k_1,k} & O \\ Y & O \\ O_{m-k-k_2,k} & O \end{bmatrix}$$

where X is $k_1 \times k$ and Y is $k_2 \times k$. Necessarily, $\rho(X) = k_1$ and $\rho(Y) = k_2$.

Proof. Let Q be the matrix such that $Q^t A^t$ is in reduced row echelon form. Then $Q^t A^t$ has all zeros in rows $k+1, ..., n$. Thus AQ has all zeros in column $k+1, ..., n$. Then $B = Q^{-1}AQ$ has all zeros in columns $k+1, ..., n$. So

$$B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix},$$

where B_1 is $k \times k$, B_2 is $(n-k) \times k$. Let P be the $k \times k$ matrix such that PB_1 is in reduced row echelon form. Let $\mathbb{R}$ be the $(n-k) \times k$ matrix such that

$$\begin{bmatrix} I_k & O \\ R & I_k \end{bmatrix} \begin{bmatrix} P & O \\ O & I_{n-k} \end{bmatrix} \begin{bmatrix} B_1 & O \\ B_2 & O \end{bmatrix} = C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} P & O \\ RP & I_{n-k} \end{bmatrix} \begin{bmatrix} B_1 & O \\ B_2 & O \end{bmatrix} = \begin{bmatrix} PB_1 & O \\ RPB_1 + B_2 & O \end{bmatrix},$$

so that if j is a pivot column of PB_1, then the j^{th} column of $RPB_1 + B_2$ has all zero entrices.

Finally, let S be the $(n-k)\times(n-k)$ matrix such that SC_2 is in reduced row echelon form. Then

$$(I_k \bigoplus S)C = \begin{bmatrix} I_k & \\ & S \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ & \vdots \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ O_{k-k_1,k} & 0 \\ D_2 & 0 \\ O_{n-k+k_2,k} & 0 \end{bmatrix},$$

where D_1 is $k_1 \times k$ and D_2 is $k_2 \times k$ for some nonnegative integers k_1 and k_2 (k_1 is the rank of B_1).

Now,

$$(I_k \bigoplus S)\begin{bmatrix} I_k & O \\ R & I_{n-k} \end{bmatrix}(P \bigoplus I_{n-k})Q^{-1}AQ(P \bigoplus I_{n-k})^{-1}\begin{bmatrix} I_k & O \\ R & I_{n-k} \end{bmatrix}^{-1}(I_k \bigoplus S)^{-1}$$

$$= D(P^{-1} \bigoplus I_{n-k})\begin{bmatrix} I_k & O \\ -R & I_{n-k} \end{bmatrix}(I_k \bigoplus S^{-1}) = \begin{bmatrix} D_1P^{-1} & O \\ O_{k-k_1,k} & O \\ D_2P^{-1} & O \\ O_{n-k-k_2,k} & O \end{bmatrix},$$

has the desired form where $X = D_1P^{-1}$ and $Y = D_2P^{-1}$. □

Lemma 4.5. *If $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ is a linear transformation which preserves $\mathfrak{M}_5$. then either T is a zero map or T is invertible.*

Proof. If T is a zero map then it is satisfied. Suppose T is not a zero map and $A \in kerT = \{A|T(A) = 0\}$ and $\rho(A) \geq \rho(Z)$ for all $Z \in kerT$.

Let $\rho(A) = k$ and $k \neq 0$.

By Lemma 4.2, $k < n$. Then

$$\begin{aligned} \rho(U^{-1}AUU^{-1}BU) + \rho(U^{-1}BUU^{-1}CU) &= \rho(U^{-1}ABU) + \rho(U^{-1}BCU) \\ &= \rho(AB) + \rho(BC) \\ &= \rho(ABC) + \rho(B) \\ &= \rho(U^{-1}AUU^{-1}BUU^{-1}CU) + \rho(U^{-1}BU) \\ &= \rho(U^{-1}ABCU) + \rho(U^{-1}BU). \end{aligned}$$

Hence $(U^{-1}AU, U^{-1}BU, U^{-1}CU) \in \mathfrak{M}_5$. Since $T(A) = U^{-1}AU$ preserves $\mathfrak{M}_5$, we have

$$U^{-1}AU = \begin{bmatrix} A_1 & A_2 & O & O \\ O & O & O & O \\ A_3 & A_4 & O & O \\ O & O & O & O \end{bmatrix}$$

by Lemma 4.3 .

Hence we may assume that

$$A = \begin{bmatrix} A_1 & A_2 \\ O & O \\ A_3 & A_4 \\ O & O \end{bmatrix},$$

where A_1 is $k_1 \times k_1, A_4$ is $k_2 \times k_2$, $k_1 + k_2 = k$ and $k + k_2 \leq n$.

Case 1, $k_1 = k$. Here

$$A = \begin{bmatrix} A_1 & O \\ O & O \end{bmatrix}.$$

Let (i, j) be a pair such that $\det A[\{1, \cdots, k\} \backslash \{i\} \,|\, \{1, \cdots, k\} \backslash \{j\}] \neq 0$.

Let $B = E_{k+1,j} + E_{i,k+1}$. Then $\rho(AB) = \rho(BA) = 1$ and $\rho(ABA) = 0$, so that $(A, B, A) \in \mathfrak{M}_5$. Thus, $T(B) = O$. But $\det(xA + B)[\{1, \cdots, k+1\} \,|\, \{1, \cdots, k+1\}]$ is a polynomial of degree $k - 1$ in x, and since $\mathbb{R}$ has at least $n + 1$ elements, for some x, $\rho(xA + B) > k$ and $T(xA + B) = O$, a contradiction to the choice of A.

Case 2, $k_1 < k$. Here

$$A = \begin{bmatrix} A_1 & A_2 & O & O \\ O & O & O & O \\ A_3 & A_4 & O & O \\ O & O & O & O \end{bmatrix}$$

and A_1 is $k_1 \times k_1$. Let $B = E_{k,k} + E_{k,k+1} + E_{k+1,k} + E_{k+1,k+1}$ and $C = E_{k,k} + E_{k,k+1} + E_{k+1,k+1}$. Then $AB = AC$ and $BA = CA$. Further $\rho(AB) = \rho(BA) = \rho(AC) = \rho(CA) = 1$, and $\rho(ABA) = \rho(ACA) \leq 1$.

If $ABA = ACA \neq O$, $(A, B, A) \in \mathfrak{M}_5$ and consequently $T(B) = O$. But, $\det(xA + B)[\{1, \cdots, k_1, k, \cdots, k + k_2\} \mid \{1, \cdots, k+1\}]$ is a polynomial in

x of degree k, and since $\mathbb{R}$ has at least $n+1$ elements, for some x, $\rho(xA+B)>k$ and $T(xA+B)=O$, a contradiction to the choice of A. If $ABA=ACA=O$, $(A,C,A)\in\mathfrak{M}_5$ and consequently $T(C)=O$. But, $\det(xA+C)[\{1,\cdots,k_1,k,\cdots,k+k_2|1,\cdots,k+1\}]$ is polynomial in x of degree k, and since $\mathbb{R}$ has at least $n+1$ elements, for some x, $\rho(xA+C)>k$ and $T(xA+C)=O$, again a contradiction to the choice of A.

Since we have reached a contradiction in each case we conclude that $k=0$ and the lemma follows. □

Lemma 4.6. *Let $T:\mathcal{M}_n(\mathbb{R})\to\mathcal{M}_n(\mathbb{R})$ and $T(X)=UXV$ for some invertible matrices U and V. Then T preserves the set $\mathfrak{M}_5$ if and only if $T(X)=\alpha PXP^{-1}$ for some invertible matrix $P\in M_n(R)$ and a non-zero real α.*

Proof. Let us consider arbitrary $(Y,Z)\in\mathfrak{M}_3$. If $\rho(Y)\le\rho(Z)$, then $\rho(YZ)=\rho(Y)$. Thus $\rho(OY)+\rho(YZ)=\rho(OYZ)+\rho(Y)$, so that $(O,Y,Z)\in\mathfrak{M}_5$. Thus $\rho(T(O)T(Y))+\rho(T(Y)T(Z))=\rho(T(O)T(Y)T(Z))+\rho(T(Y))$. That is $\rho(T(Y)T(Z))=\rho(T(Y))$, and since $T(X)=UXV$, we have $\rho(T(Y))\le\rho(T(Z))$. Thus, $(T(Y),T(Z))\in\mathfrak{M}_3$. If $\rho(Z)\le\rho(Y),(Y,Z,O)\in\mathfrak{M}_5$, and similar to the above argument, $(T(Y),T(Z))\in\mathfrak{M}_3$. Thus, T preserves $\mathfrak{M}_3$. By Theorem 2.3 the lemma follows. □

We obtain the characterization of linear operators that preserve $\mathfrak{M}_5$.

Theorem 4.7. *Let $T:\mathcal{M}_n(\mathbb{R})\to\mathcal{M}_n(\mathbb{R})$ be a bijective linear transformation. Then T preserves the set $\mathfrak{M}_5$ if and only if $T(X)=\alpha PXP^{-1}$ for some invertible matrix $P\in\mathcal{M}_n(\mathbb{R})$ and non-zero real α.*

Proof. ($\Longleftarrow$) If $T(X)=\alpha PXP^{-1}$ for some invertible $P\in\mathcal{M}_n(\mathbb{R})$, and $\alpha\neq 0$, then T preserves $\mathfrak{M}_5$. by the proof in Lemma 4.6.

($\Longrightarrow$) By Lemma 4.2 T preserves the set of nonsingular matrices. By Lemma 1.1, T is a (U,V)-operator. Therefore T is a composition of operators of the form

$T(X)=UXV$ for some U and V nonsingular; or

$T(X)=X^t$ where X^tdenotes the transpose of X.

Suppose $T(X)=X^t$ and $J_j=I-E_{j,j}$. Then $E_{ij}IJ_j=0$ but $E_{ji}IJ_j=E_{ji}$. Thus $(E_{i,j},I,J_j)\in\mathfrak{M}_5$ but $(E_{j,i},I,I_j)\notin\mathfrak{M}_5$. Hence $T(X)=X^t$ dose not preserve $\mathfrak{M}_5$.

Therefore $T(X) = UXV$ for some invertible U, V. By Lemma 4.6, $T(X) = \alpha PXP^{-1}$ for some invertible matrix P and nonzero real α. □

Thus in this section, we obtained the characterizations of the linear operator that preserve the set of matrix triples in the extreme rank of three matrix product.

References

[1] L. B. Beasley, Linear operators which preserve pairs on which the rank is additive, J. Korean S. I. A. M.,2(1998) 27-30.

[2] L. B. Beasley, A. E. Guterman, and C. L. Neal, Linear preservers for Sylvester and Frobenius bounds on matrix rank, Rocky Mountains J. Math. 36(1)(2006), 67-75.

[3] L. B. Beasley, A. E. Guterman, Y. B. Jun and S. Z. Song, Linear preservers of extremes of rank inequalities over semirings: Row and Column ranks, Linear Algebra Appl., 413(2006), 495-509.

[4] L. B. Beasley and T. L. Laffey, Linerar operators on matrices: The invariance of rank-k matrices, Linerar Algebra Appl., 133(1990), 175-184.

[5] L. B. Beasley, S.-G. Lee, and S.-Z. Song, Linear operators that preserve pairs of matrices which satisfy extreme rank properties, Linear ALgebra Appl. 350 (2002), 263-272.

[6] L. B. Beasley, S.-G. Lee, S.-Z. Song, Linear operators that preserve zero-term rank of Boolean matrices, *J. Korean Math. Soc.*, V. 36, no. 6, 1999, pp. 1181-1190.

[7] P. Pierce and others, A Survey of Linear Preserver Problems, *Linear and Multilinear Algebra*, 33 (1992) 1-119.

[8] S. Z. Song, Topics on linear preserver problems - a brief introduction (Korean), *Commun. Korean Math. Soc.*, 21(2006), 595-612.

〈국문초록〉

실수상의 행렬 곱들의 계수에 대한 극치집합 보존자

본 논문에서는 실수상의 행렬의 짝들로 구성되는 집합들을 구성하였다. 이 행렬짝들의 집합은 행렬들의 곱에서 그 계수를 생각할 때 ,그 곱의 계수는 각각의 행렬들의 계수와 부등식 관계를 형성하는데서 착안하여, 이 부등식이 등식이 되는 극치인 경우들에서 자연스럽게 나타나는 집합들이다. 이 행렬 짝들의 집합들을 실수 상의 행렬들에서 부등식들의 극치인 경우들로 구성하였다. 실수 상의 행렬 A 의 계수를 $\rho(A)$로 표시한다. 그러면, n차 정방 행렬 A 와 B, C 에 대하여, 다음과 같은 행렬 짝들의 집합을 구성할 수 있다;

$\sum_1 = \{(A,B)|\rho(A+B) = \rho(A)+\rho(B)\}$;

$\sum_2 = \{(A,B)|\rho(A+B) = |\rho(A)-\rho(B)|\}$;

$\mathfrak{M}_3 = \{(A,B)|\rho(AB) = \min\{\rho(A),\rho(B)\}\}$;

$\mathfrak{M}_4 = \{(A,B)|\rho(AB) = \rho(A)+\rho(B)-n\}$;

$\mathfrak{M}_5 = \{(A,B,C)|\rho(AB)+\rho(BC) = \rho(ABC)+\rho(B)\}$;

이상의 실수상의 행렬 짝들의 집합을 선형연산자로 보내어 그 집합의 성질들을 보존하는 선형연산자를 연구하여 그 형태를 규명하였다. 곧, 이러한 실수 행렬 짝들의 집합을 보존하는 선형연산자를 $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ 라 둘 때 그의 형태는 $T(X) = \alpha PXP^{-1}$ 로 나타남을 보이고, 이들을 증명하였다. 그리고 이 선형연산자가 위의 집합들을 보존함을 증명하였다.

감사의 글

어느덧 짧지 않은 대학원 생활을 마무리하며 지난 시간들을 돌이켜 보니 즐겁고 행복했던것 같습니다. 저에게 많은 도움을 주신 교수님들과 선배님들 덕분에 대학원 생활을 마치면서 이렇게 논문을 잘 마무리 하게 되어 기쁩니다.

처음 대학원에 입학했을때에는 설레임반 두려움반이였습니다. 학부시절 공부를 열심히 못한것이 아쉬워 대학원 진학을 결심했지만 막상 수업을 시작하고 보니 학부시절과는 또 다른 수업과정에 의해 많은 어려움을 겪었습니다. 중간에 포기하고 싶을때도 있었지만 졸업논문이 나오기까지 제가 이렇게 성장할수 있도록 힘이되고 방향을 잡아주셨던 많은 분들께 감사의 말씀을 전하고자 합니다.

먼저 본 논문이 완성되기까지 세심한 지도와 많은 격려로 이끌어 주신 송석준 교수님께 진심으로 감사드립니다. 또한 아낌없는 지도로 많은 가르침을 주신 방은숙 교수님, 양영오 교수님, 정승달 교수님, 윤용식 교수님, 유상욱 교수님, 진현성 교수님 께도 감사드리며 언제나 옆에서 도움을 주신 강경태 선생님과 이지순 선생님께도 감사드립니다.

수학이라는 폭넓은 학문에 좀더 다가갈수 있도록 옆에서 많은 도움을 준 민주언니, 금란언니, 연정언니, 은아언니 에게도 고마움을 전합니다. 특히 함께 입학하여 서로 연구하는데 많은 도움과 힘이 되어준 혜정언니에게도 고맙다는 말을 전하고 싶습니다.

마지막으로 항상 사랑으로 키워주시고 부족한 자식을 믿어주신 부모님께 감사의 말씀을 드립니다. 언제나 제 편이 되어 힘을 주시고 바르게 생각하고 행동할수 있도록 가르쳐주신 부모님께 누가 되지 않는 딸이 되고 더욱 성장하는 딸이 되도록 노력하겠습니다. 또한 언제나 묵묵히 옆에서 지켜봐주고 응원을 해준 하나뿐인 남동생 강현이에게 고맙다는 말을 전하고 싶습니다.

그리고 여기서 거론하지 못하지만 도움을 주신 여러 분 들이 있는데, 마음으로 감사의 인사를 전합니다. 대학원 과정을 마치고 새로운 생활을 목전에 둔 상황에서 미래가 불확실 하지만 지금까지 저에게 많은 도움과 가르침을 주신 분들의 지도편달을 받으며, 더욱 꿋꿋이 제 앞길을 개척해 나가겠습니다.

2009년 12월