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Zadeh's extension principle for two non-positive triangular fuzzy numbers

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Hyun Kim

(Supervised by professor Yong Sik Yun)

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指導敎授 尹 龍 植

金 炫

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ZADEH'S EXTENSION PRINCIPLE FOR TWO NON-POSITIVE TRIANGULAR FUZZY NUMBERS

Hyun Kim

Abstract. There are many results for fuzzy expansion operations based on Zadeh's expansion principle. In particular, many results of extended algebraic operations between two triangular fuzzy numbers are well known. We calculate a max-min composition operator for two non-positive triangular fuzzy numbers.

1. Introduction

Traditional logic is either true or false. Which element belongs to which set? Or does not belong? Dichotomous set theory actually represents only a part of our lives. In order to express the characteristics of uncertain and complex life better, we need a more extended logic concept and an extended proposition concept. Fuzzification plays an important role in solving this problem. So as a solution to this problem, Zadeh introduced a fuzzy set in the early 1960s. It is not limited to two cases where an element belongs to a particular set or does not belong to a particular set. The term "membership function" is used to denote a real number corresponding to the degree of truth that falls between the elements of the closed interval [0, 1] ([2], [5], [6]). In fuzzy set theory, various types of operations between two fuzzy sets have been defined and studied. Consider the case where several variables are mutually combined by some function or relationship. At this time, Zadeh introduced the principle of expansion on how to define the output variable when the fuzzy set is defined for each variable and the relationship between input and output being clearly given. There are many results for fuzzy expansion operations based on Zadeh's expansion principle. In particular, many results of extended algebraic operations between two triangular fuzzy numbers are well known ([1], [3], [4]).

In this paper, we calculate a max-min composition operator for two non-positive triangular fuzzy numbers.

2. Preliminaries

Let X be a set. A classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0, 1\}$ such that $\mu_A(x) = 1$ if $x \in A$, and $\mu_A(x) = 0$ if $x \notin A$. $\{0, 1\}$ is called a valuation set. The following definition is a generalization of this notion.

Definition 2.1. A fuzzy set A on X is a function from X to the interval $[0, 1]$. The function is called the *membership function* of A.

Key words and phrases: max-min compositional operator, non-positive triangular fuzzy numbers.

Let A be a fuzzy set on X with a membership function μ_A. Then A is a subset of X that has no sharp boundary. A is completely characterized by the set of pairs $A = \{(x, \mu_A(x)), x \in X\}$ elements with a zero degree of membership are normally not listed.

Definition 2.2. The set $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$ is said to be the α-cut of a fuzzy set A.

Definition 2.3. A fuzzy set A on $\mathbb{R}$ is *convex* if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \quad \forall x_1, x_2 \in \mathbb{R}, \quad \forall \lambda \in [0,1]$.

Definition 2.4. A convex fuzzy set A on $\mathbb{R}$ is called a *fuzzy number* if

(1) There exists exactly one $x \in \mathbb{R}$ such that $\mu_A(x) = 1$,
(2) $\mu_A(x)$ is piecewise continuous.

Definition 2.5. A triangular fuzzy number on $\mathbb{R}$ is a fuzzy number A which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, \quad a_3 \leq x, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

The above triangular fuzzy number is denoted by $A = (a_1, a_2, a_3)$.

Definition 2.6. *The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as*

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

Remark 2.7. Let A and B be fuzzy sets and $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B, A(-)B, A(\cdot)B$ and $A(/)B$ can be calculated as follows.
(1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$.
(2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$.
(3) $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [\min(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}),$

$$\max(a_1^{(\alpha)}b_1^{(\alpha)}, a_1^{(\alpha)}b_2^{(\alpha)}, a_2^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)})].$$

(4) $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [\min(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)}),$
$$\max(a_1^{(\alpha)}/b_1^{(\alpha)}, a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}, a_2^{(\alpha)}/b_2^{(\alpha)})].$$

EXAMPLE 2.8. *For two triangular fuzzy numbers $A = (1, 2, 4)$ and $B = (2, 4, 5)$, we have*

1. *Addition :* $A(+)B = (3, 6, 9)$.
2. *Subtraction :* $A(-)B = (-4, -2, 2)$.
3. *Multiplication :*

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < 2, \ \ 20 \le x, \\ \frac{-2+\sqrt{2x}}{2}, & 2 \le x < 8, \\ \frac{7-\sqrt{9+2x}}{2}, & 8 \le x < 20. \end{cases}$$

Note that $A(\cdot)B$ is not a triangular fuzzy number.

4. *Division :*

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < \frac{1}{5}, \ \ 2 \le x, \\ \frac{5x-1}{x+1}, & \frac{1}{5} \le x < \frac{1}{2}, \\ \frac{-x+2}{x+1}, & \frac{1}{2} \le x < 2. \end{cases}$$

Note that $A(/)B$ is not a triangular fuzzy number.

3. Main results

Let a, b, c, p, q, r be six positive real numbers. Consider two triangular fuzzy numbers $A = (-a, -b, c)$ and $B = (-p, q, r)$. Let $-a < -p$ and $a_2^{(\alpha_1)} = b_1^{(\alpha_2)} = 0$. The results of a max-min composition operator for two non-positive triangular fuzzy numbers A and B are divided into 8 cases.

CASE 1 : $\alpha_1 > \alpha_2$ and $c > r$

Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \ \ c < x, \\ \frac{1}{a-b}(x+a), & -a \le x < -b, \\ \frac{1}{c+b}(c-x) & -b \le x \le c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \ \ r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(r-x) & q \le x \le r. \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x)=0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b)=1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x \le c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :

By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)}-b_2^{(\alpha)}, a_2^{(\alpha)}-b_1^{(\alpha)}] = [\alpha(a-b)-a+\alpha(r-q)-r, -\alpha(c+b)+c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x)=0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q)=1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(-bq)=1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x < -bq, \\ \frac{bp+cp+bq+cq+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+bq+cq)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x \le (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p). \end{cases}$$

(2) $\alpha_2 < \alpha \leq \alpha_1$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r) \leq x \\ \qquad < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)} \\ (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \leq x \\ \qquad \leq (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r). \end{cases}$$

(3) $0 < \alpha \leq \alpha_2$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \leq x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ \qquad (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p) \leq x < cr. \end{cases}$$

There is a case like this, or the following. By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)},\\ \qquad -ar\le x<(\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r),\\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)},\\ \qquad (\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p)\le x\le ap.\end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1<\alpha\le 1$

By the above facts, $A_\alpha(/)B_\alpha=\left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left[\frac{\alpha(a-b)-a}{\alpha(q+p)-p},\frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$ and $\mu_{A(/)B}(x)=\alpha_1$ at $x=\frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $x=\frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}$, $\mu_{A(/)B}(\frac{-b}{q})=1$. Therefore,

$$\mu_{A(/)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)}\le x\le\frac{-b}{q},\\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q}\le x\le\frac{(-\alpha_1(c+b)+c)}{(-\alpha_1(r-q)+r)}.\end{cases}$$

(2) $\alpha_2<\alpha\le\alpha_1$

By the above facts, $A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right)=\left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p},\frac{-\alpha(c+b)+c}{\alpha(q+p)-p}\right)$ and $\mu_{A(/)B}(x)=\alpha_1$ at $x=\frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $x=\frac{-\alpha_1(c+b)+c}{\alpha_1(q+p)-p}$. Therefore

$$\mu_{A(/)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & -\infty<x\le\frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)},\\ \frac{px+c}{(q+p)x+c+b}, & \frac{(-\alpha_1(c+b)+c)}{(\alpha_1(q+p)-p)}\le x<\infty.\end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

EXAMPLE 3.1. Let $A=(-5,-4,6)$ and $B=(-1,2,4)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(0)}>0,\quad b_2^{(0)}<0,\quad b^{(1)}>0,\quad b^{(\alpha_2)}=0,\quad \alpha_1>\alpha_2,\quad a_2^{(0)}>b_2^{(0)}$$

For

$$\mu_A(x)=\begin{cases}0, & x<-5,\ 6<x,\\ x+5, & -5\le x<-4,\\ -\frac{1}{10}x+\frac{3}{5}, & -4\le x\le 6,\end{cases}$$

and

$$\mu_B(x)=\begin{cases}0, & x<-1,\ 4<x,\\ \frac{1}{3}x+\frac{1}{3}, & -1\le x<2,\\ -\frac{1}{2}x+2, & 2\le x\le 4,\end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 5$ and $\alpha = -\frac{a_2^{(\alpha)}}{10} + \frac{3}{5}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 5, -10\alpha + 6]$. Since $\alpha = \frac{b_1^{(\alpha)}}{3} + \frac{1}{3}$ and $\alpha = -\frac{b_2^{(\alpha)}}{2} + 2$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [3\alpha - 1, -2\alpha + 4]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [4\alpha - 6, -12\alpha + 10]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-6, 10]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -2$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -6, \quad 10 < x, \\ \frac{1}{4}x + \frac{3}{2}, & -6 \le x < -2, \\ -\frac{1}{12}x + \frac{5}{6}, & -2 \le x \le 10, \end{cases}$$

i.e., $A(+)B = (-6, -2, 10)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [3\alpha - 9, -13\alpha + 7]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-9, 7]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -6$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -9, \quad 7 < x, \\ \frac{1}{3}x + 3, & -9 \le x < -6, \\ -\frac{1}{13}x + \frac{7}{13}, & -6 \le x \le 7, \end{cases}$$

i.e., $A(-)B = (-9, -6, 7)$.

(3) Multiplication : (i) $\frac{3}{5} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, -30\alpha^2 + 28\alpha - 6]$, $\mu_{A(\cdot)B}(-\frac{308}{25}) = \frac{3}{5}$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -8$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 2x}}{2}, & -\frac{308}{25} \le x < -8, \\ \frac{14 + \sqrt{16 - 30x}}{30}, & -8 \le x < 0. \end{cases}$$

(ii) $\frac{1}{3} \le \alpha \le \frac{3}{5}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, 20\alpha^2 - 52\alpha + 24]$, $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -\frac{140}{9}, \frac{80}{3}$ and $\mu_{A(\cdot)B}(x) = \frac{3}{5}$ at $x = -\frac{308}{25}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 2x}}{2}, & -\frac{140}{9} \le x < -\frac{308}{25}, \\ \frac{13 - \sqrt{49 + 5x}}{10}, & 0 \le x < \frac{80}{3}. \end{cases}$$

(iii) $0 \le \alpha \le \frac{1}{3}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, 20\alpha^2 - 52\alpha + 24]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-20, 24]^c$ and $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -\frac{140}{9}, \frac{80}{3}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7-\sqrt{9-2x}}{2}, & -20 \le x < -\frac{140}{9}, \\ \frac{13-\sqrt{49+5x}}{10}, & \frac{80}{3} \le x < 24. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{6}{10} < x \le 1$

Since $A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left[\frac{\alpha-5}{3\alpha-1}, \frac{-10\alpha+6}{-2\alpha+4}\right]$, $\mu_{A(/)B}(x) = \frac{6}{10}$ at $x = -\frac{11}{2}, 0$ and $\mu_{A(/)B}(x) = 1$ at $x = -2$.

(ii) $\frac{1}{3} < x < \frac{6}{10}$

Since $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right) = \left(\frac{\alpha-5}{3\alpha-1}, \frac{-10\alpha+6}{3\alpha-1}\right)$, by the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{x-5}{3x-1}, & -\infty < x < -\frac{11}{2}, \\ \frac{x-5}{3x-1}, & -\frac{11}{2} \le x < -2, \\ \frac{2x-3}{x-5}, & -2 \le x < 0, \\ \frac{x+6}{3x+10}, & 0 \le x < \infty. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 2 : $\alpha_1 = \alpha_2$ and $c > r$

Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \ \ c < x, \\ \frac{1}{a-b}(x+a), & -a \le x < -b, \\ \frac{1}{c+b}(c-x), & -b \le x \le c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \ \ r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(q-x), & q \le x \le r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q+p) - p, -\alpha(c+b) + c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x \le c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :
By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x < (-\alpha_1(c+b) + c) \cdot (\alpha_1(q+p) - p). \end{cases}$$

(2) $0 < \alpha \le \alpha_1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), (-\alpha(c+b) + c) \cdot (-\alpha(r-q) + r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (-\alpha_1(c+b) + c) \cdot (-\alpha_1(r-q) + r)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar\le x<(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)\le x<cr.\end{cases}$$

There is a case like this, or the following. By the above fact,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_1^{(\alpha)}\cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=0$ on the interval $[-ar,ap]^c$ and $\mu_{A(\cdot)B}(x)=\alpha_1$ at $x=(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x=(\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar\le x<(\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)\le x<ap.\end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1<\alpha\le 1$

By the above fact, $A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(-\alpha(c+b)+c)}{(-\alpha(r-q)+r)}\right]$ and $\mu_{A(/)B}(x)=1$ at $x=\frac{-b}{q}$. Therefore,

$$\mu_{A(/)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & -\infty<x\le\frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q}\le x\le 0.\end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

Example 3.2. Let $A=(-5,-4,4)$ and $B=(-1,1,2)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(0)}<0,\quad b_2^{(0)}>0,\quad b^{(1)}>0,\quad b^{(\alpha_2)}=0,\quad \alpha_1=\alpha_2,\quad a_2^{(0)}>b_2^{(0)}$$

For

$$\mu_A(x)=\begin{cases}0, & x<-5,\ 4<x, \\ x+5, & -5\le x<-4, \\ -\frac{1}{8}x+\frac{1}{2}, & -4\le x\le 4,\end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -1, \ \ 2 < x, \\ \frac{1}{2}x + \frac{1}{2}, & -1 \le x < 1, \\ -x + 2, & 1 \le x \le 2, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 5$ and $\alpha = -\frac{a_2^{(\alpha)}}{8} + \frac{1}{2}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 5, -8\alpha + 4]$. Since $\alpha = \frac{b_1^{(\alpha)}}{2} + \frac{1}{2}$ and $\alpha = -b_2^{(\alpha)} + 2$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [2\alpha - 1, -\alpha + 2]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [3\alpha - 6, -9\alpha + 6]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-6, 6]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -3$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -6, \ \ 6 < x, \\ \frac{1}{3}x + 2, & -6 \le x < -3, \\ -\frac{1}{9}x + \frac{2}{3}, & -3 \le x \le 6, \end{cases}$$

i.e., $A(+)B = (-6, -3, 6)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 7, -10\alpha + 5]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7, 5]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -5$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7, \ \ 5 < x, \\ \frac{1}{2}x + \frac{7}{2}, & -7 \le x < -5, \\ -\frac{1}{10}x + \frac{1}{2}, & -5 \le x \le 5, \end{cases}$$

i.e., $A(-)B = (-4, -2, 2)$.

(3) Multiplication : (i) $\frac{1}{2} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-\alpha^2 + 7\alpha - 10, -16\alpha^2 + 16\alpha - 4]$, $\mu_{A(\cdot)B}(x) = \frac{1}{2}$ at $x = -\frac{27}{4}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 4x}}{2}, & -\frac{27}{4} \le x < -4, \\ \frac{2 + \sqrt{-x}}{4}, & -4 \le x < 0. \end{cases}$$

(ii) $0 \le \alpha \le \frac{1}{2}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, 8\alpha^2 - 20\alpha + 8]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-20, 8]^c$ and $\mu_{A(\cdot)B}(x) = \frac{1}{2}$ at $x = -\frac{27}{4}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 2x}}{2}, & -20 \le x < -\frac{27}{4}, \\ \frac{5 - \sqrt{9 + 2x}}{4}, & 0 \le x < 8. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{1}{2} < x \leq 1$

Since $A_\alpha(/)B_\alpha = (\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}] = (\frac{\alpha-5}{2\alpha-1}, \frac{-8\alpha+4}{-\alpha+2}]$, $\mu_{A(/)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{x-5}{2x-1}, & -\infty < x < -4, \\ \frac{2x-4}{x-8}, & -4 \leq x < 0. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 3 : $\alpha_1 < \alpha_2$ and $c > r$

Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \ \ c < x, \\ \frac{1}{a-b}(x+a), & -a \leq x < -b, \\ \frac{1}{c+b}(c-x) & -b \leq x \leq c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \ \ r < x, \\ \frac{1}{q+p}(x+p), & -p \leq x < q, \\ \frac{1}{r-q}(r-x) & q \leq x \leq r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively, Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \ \ -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \leq x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \leq x \leq c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :

By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r-q) - r, -\alpha(c+b) + c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p,\ \ -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), (-\alpha(c+b) + c) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = \alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r)$ and $x = (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+be+ce)}, \\ \qquad -bq \le x < (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p). \end{cases}$$

(2) $\alpha_1 < \alpha \le \alpha_2$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b) - a) \cdot (-\alpha(r-q) + r), (\alpha(a-b) - a) \cdot (\alpha(q+p) - p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r)$ and $x = (\alpha_1(a-b) - a) \cdot (\alpha_1(q+p) - p)$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r)$ and $x = (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \alpha_1(a-b) - a) \cdot (-\alpha_1(r-q) + r) \le x \\ \qquad < (\alpha_2(a-b) - a) \cdot (-\alpha_2(r-q) + r), \\ \frac{2ap+aq-bp+\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ (\alpha_1(a-b) - a) \cdot (\alpha_1(q+p) - p) \le x \\ \qquad < (-\alpha_2(c+b) + c) \cdot (\alpha_2(q+p) - p). \end{cases}$$

(3) $0 < \alpha \leq \alpha_1$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \leq x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \leq x < cr. \end{cases}$$

There is a case like this, or the following. By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \leq x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \leq x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1 < \alpha \leq 1$

By the above fact, $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(-\alpha(c+b)+c)}{(-\alpha(r-q)+r)}\right]$ and $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \leq \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \leq x \leq 0. \end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

Example 3.3. Let $A = (-6, -5, 5)$ and $B = (-2, 1, 3)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(0)} < 0, \quad b_2^{(0)} > 0, \quad b^{(1)} > 0, \quad b^{(\alpha_2)} = 0, \quad \alpha_1 < \alpha_2, \quad a_2^{(0)} > b_2^{(0)}$$

For

$$\mu_A(x) = \begin{cases} 0, & x < -6, \ 5 < x, \\ x + 6, & -6 \le x < -5, \\ -\frac{1}{10}x + \frac{1}{2}, & -5 \le x \le 5, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -2, \ 3 < x, \\ \frac{1}{3}x + \frac{2}{3}, & -2 \le x < 1, \\ -\frac{1}{2}x + \frac{3}{2}, & 1 \le x \le 3, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)}, \ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, \ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 6$ and $\alpha = -\frac{a_2^{(\alpha)}}{10} + \frac{1}{2}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 6, -10\alpha + 5]$. Since $\alpha = \frac{b_1^{(\alpha)}}{3} + \frac{2}{3}$ and $\alpha = -\frac{b_2^{(\alpha)}}{2} + \frac{3}{2}$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [3\alpha - 2, -2\alpha + 3]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [4\alpha - 8, -12\alpha + 8]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-8, 8]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -1$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -8, \ 8 < x, \\ \frac{1}{4}x + 2, & -8 \le x < -1, \\ -\frac{1}{12}x + \frac{2}{3}, & -1 \le x \le 8, \end{cases}$$

i.e., $A(+)B = (-8, -1, 8)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [3\alpha - 9, -13\alpha + 7]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-9, 7]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -6$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -9, \ -6 < x, \\ \frac{1}{3}x + 3, & -9 \le x < -6, \\ -\frac{1}{13}x + \frac{7}{13}, & -6 \le x \le 7, \end{cases}$$

i.e., $A(-)B = (-9, -6, 7)$.

(3) Multiplication : (i) $\frac{2}{3} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-2\alpha^2 + 15\alpha - 18, -30\alpha^2 + 35\alpha - 10]$, $\mu_{A(\cdot)B}(x) = \frac{2}{3}$ at $x = -\frac{80}{9}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -5$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{15-\sqrt{81-8x}}{4}, & -\frac{80}{9}\le x\le -5,\\ \frac{35+\sqrt{25-120x}}{60}, & -5\le x\le 0.\end{cases}$$

(ii) $\frac{1}{2}\le\alpha\le\frac{2}{3}$

Since $A_\alpha(\cdot)B_\alpha=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_1^{(\alpha)}\cdot b_1^{(\alpha)}]=[-2\alpha^2+15\alpha-18,3\alpha^2-20\alpha+12]$, $\mu_{A(\cdot)B}(x)=\frac{1}{2}$ at $x=-11,\frac{11}{4}$, and $\mu_{A(\cdot)B}(x)=\frac{2}{3}$ at $x=-\frac{80}{9},0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{15-\sqrt{81-8x}}{4}, & -11\le x\le -\frac{80}{9},\\ \frac{10-\sqrt{64+3x}}{3}, & 0\le x\le \frac{11}{4}.\end{cases}$$

(iii) $\frac{3}{17}\le\alpha\le\frac{1}{2}$

Since $A_\alpha(\cdot)B_\alpha=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_1^{(\alpha)}\cdot b_1^{(\alpha)}]=[-2\alpha^2+15\alpha-18,3\alpha^2-20\alpha+12]$, $\mu_{A(\cdot)B}(x)=\frac{3}{17}$ at $x=-\frac{4455}{289},\frac{2421}{289}$ and $\mu_{A(\cdot)B}(x)=\frac{1}{2}$ at $x=-11,\frac{11}{4}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{15-\sqrt{81-8x}}{4}, & -\frac{4455}{289}\le x\le -11,\\ \frac{10-\sqrt{64+3x}}{3}, & \frac{11}{4}\le x\le \frac{2421}{289}.\end{cases}$$

(iv) $0\le\alpha\le\frac{3}{17}$

Since $A_\alpha(\cdot)B_\alpha=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_2^{(\alpha)}\cdot b_2^{(\alpha)}]=[-2\alpha^2+15\alpha-18,20\alpha^2-40\alpha+15]$, $\mu_{A(\cdot)B}(x)=0$ on the interval $[-18,15]^c$ and $\mu_{A(\cdot)B}(x)=\frac{3}{17}$ at $x=-\frac{4455}{289},\frac{2421}{289}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{15-\sqrt{81-8x}}{4}, & -18\le x\le -\frac{4455}{289},\\ \frac{13-\sqrt{49+5x}}{10}, & \frac{2421}{289}\le x\le 15.\end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{2}{3}<x\le 1$

Since $A_\alpha(/)B_\alpha=\big(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\big]=\big(\frac{\alpha-6}{3\alpha-2},\frac{-10\alpha+5}{-2\alpha+3}\big]$, $\mu_{A(/)B}(x)=1$ at $x=-5$. By the routine calculation, we have

$$\mu_{A(/)B}(x)=\begin{cases}\frac{2x-6}{3x-1}, & -\infty<x<-5,\\ \frac{3x-5}{2x-10}, & -5\le x\le -1.\end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 4 : $\alpha_1>\alpha_2$ and $c<r$

Note that

$$\mu_A(x)=\begin{cases}0, & x<-a,\ \ c<x,\\ \frac{1}{a-b}(x+a), & -a\le x<-b,\\ \frac{1}{c+b}(c-x) & -b\le x\le c,\end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \quad r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(r-x) & q \le x \le r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x \le c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :

By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b)-a+\alpha(r-q)-r, -\alpha(c+b)+c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$

By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(-bq) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x < -bq, \\ \frac{bp+cp+bq+cq+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+bq+cq)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x < (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p). \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_2^{(\alpha)}\cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r) \le x \\ \qquad < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \le x \\ \qquad < (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r). \end{cases}$$

(3) $0 < \alpha \le \alpha_2$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_2^{(\alpha)}\cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(-\alpha_2(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{-cq+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+br+cr)(-cr+x)}}{2(-bq-cq+br+cr)}, \\ \qquad (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p) \le x < cr. \end{cases}$$

There is a case like this, or the following. By the above facts,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $(\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-rb)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_2(a-b)-a)\cdot(\alpha_2(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1 < \alpha \le 1$

By the above facts, $A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left[\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{-\alpha(r-q)+r}\right]$ and $\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{\alpha_1(a-b)-a}{\alpha_1(q+p)-p}$ and $x = \frac{-\alpha_1(c+b)+c}{-\alpha_1(r-q)+r}$, $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)} \le x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le \frac{(-\alpha_1(c+b)+c)}{(-\alpha_1(r-q)+r)}. \end{cases}$$

(2) $\alpha_2 < \alpha \le \alpha_1$

By the above facts, $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right) = \left(\frac{\alpha(a-b)-a}{\alpha(q+p)-p}, \frac{-\alpha(c+b)+c}{\alpha(q+p)-p}\right)$ and $\mu_{A(/)B}(x) = \alpha_1$ at $x = \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)}$ and $x = \frac{(-\alpha_1(c+b)+c)}{(\alpha_1(q+p)-p)}$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \le \frac{(\alpha_1(a-b)-a)}{(\alpha_1(q+p)-p)}, \\ \frac{px+c}{(q+p)x+c+b}, & \frac{(-\alpha_1(c+b)+c)}{(\alpha_1(q+p)-p)} \le x < \infty. \end{cases}$$

EXAMPLE 3.4. Let $A = (-4, -3, 3)$ and $B = (-1, 2, 4)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(0)} > 0, \quad b_2^{(0)} < 0, \quad b^{(1)} > 0, \quad b^{(\alpha_2)} = 0, \quad \alpha_1 > \alpha_2, \quad a_2^{(0)} < b_2^{(0)}$$

For

$$\mu_A(x) = \begin{cases} 0, & x < -4, \ 3 < x, \\ x+4, & -4 \le x < -3, \\ -\frac{1}{6}x + \frac{1}{2}, & -3 \le x \le 3, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -1, \ \ 4 < x, \\ \frac{1}{3}x + \frac{1}{3}, & -1 \le x < 2, \\ -\frac{1}{2}x + 2, & 2 \le x \le 4, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 4$ and $\alpha = -\frac{a_2^{(\alpha)}}{6} + \frac{1}{2}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 4, -6\alpha + 3]$. Since $\alpha = \frac{b_1^{(\alpha)}}{3} + \frac{1}{3}$ and $\alpha = -\frac{b_2^{(\alpha)}}{2} + 2$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [3\alpha - 1, -2\alpha + 4]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [4\alpha - 5, -8\alpha + 7]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-5, 7]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -1$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -5, \ \ 7 < x, \\ \frac{1}{4}x + \frac{5}{4}, & -5 \le x < -1, \\ -\frac{1}{8}x + \frac{7}{8}, & -1 \le x \le 7, \end{cases}$$

i.e., $A(+)B = (-5, -1, 7)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [3\alpha - 8, -9\alpha + 4]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-8, 4]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -5$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -8, \ \ 4 < x, \\ \frac{1}{3}x + \frac{8}{3}, & -8 \le x < -5, \\ -\frac{1}{9}x + \frac{4}{9}, & -5 \le x \le 4, \end{cases}$$

i.e., $A(-)B = (-8, -5, 4)$.

(3) Multiplication : (i) $\frac{1}{2} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, -30\alpha^2 + 28\alpha - 6]$, $\mu_{A(\cdot)B}(x) = \frac{3}{5}$ at $x = -\frac{308}{25}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -8$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 2x}}{2}, & -\frac{308}{25} \le x \le -8, \\ \frac{14 + \sqrt{16 - 30x}}{30}, & -8 \le x \le 0. \end{cases}$$

(ii) $\frac{1}{3} \le \alpha \le \frac{1}{2}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, 20\alpha^2 - 52\alpha + 24]$, $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -\frac{140}{9}, \frac{80}{3}$ and $\mu_{A(\cdot)B}(x) = \frac{3}{5}$ at $x = -\frac{308}{25}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 2x}}{2}, & -\frac{140}{9} \le x \le -\frac{308}{25}, \\ \frac{13 - \sqrt{49 + 5x}}{10}, & 0 \le x \le \frac{80}{3}. \end{cases}$$

(iii) $0 \le \alpha \le \frac{1}{3}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-2\alpha^2 + 14\alpha - 20, 20\alpha^2 - 52\alpha + 24]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-20, 24]^c$ and $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -\frac{140}{9}, \frac{80}{3}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7-\sqrt{9-2x}}{2}, & -20 \le x \le -\frac{140}{9}, \\ \frac{13-\sqrt{49+5x}}{10}, & \frac{80}{3} \le x \le 24. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{1}{2} < x \le 1$

Since $A_\alpha(/)B_\alpha = \left[\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left[\frac{\alpha-4}{3\alpha-1}, \frac{-6\alpha+3}{-2\alpha+4}\right]$, $\mu_{A(/)B}(x) = \frac{1}{2}$ at $x = -7, 0$ and $\mu_{A(/)B}(x) = 1$ at $x = -\frac{3}{2}$.

(ii) $\frac{1}{3} < x < \frac{1}{2}$

Since $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right) = \left(\frac{\alpha-4}{3\alpha-1}, \frac{-6\alpha+3}{3\alpha-1}\right)$, $\mu_{A(/)B}(x) = \frac{1}{2}$ at $x = -7, 0$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{x-4}{3x-1}, & -\infty < x < -7, \\ \frac{x-4}{3x-1}, & -7 \le x < -\frac{3}{2}, \\ \frac{4x-3}{2x-6}, & -\frac{3}{2} \le x < 0, \\ \frac{x+3}{3x+6}, & 0 \le x < \infty. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 5 : $\alpha_1 = \alpha_2$ and $c < r$

Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \quad c < x, \\ \frac{1}{a-b}(x+a), & -a \le x < -b, \\ \frac{1}{c+b}(c-x) & -b \le x \le c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \quad r < x, \\ \frac{1}{q+p}(x+p), & -p \le x < q, \\ \frac{1}{r-q}(r-x) & q \le x \le r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b) - a, -\alpha(c+b) + c].$$

Similarly,

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :
By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x \le c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :
By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b)-a+\alpha(r-q)-r, -\alpha(c+b)+c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_1 < \alpha \le 1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+bq+cq)}, \\ \qquad -bq \le x < (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p). \end{cases}$$

(2) $0 < \alpha \le \alpha_1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] \\ &= [\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \le x < cr. \end{cases}$$

There is a case like this, or the following. By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_1^{(\alpha)}\cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1 < \alpha \le 1$

By the above fact, $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(-\alpha(c+b)+c)}{(-\alpha(r-q)+r)}\right]$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{-b}{q}$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \le \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \le x \le 0. \end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

Example 3.5. Let $A = (-4, -3, 1)$ and $B = (-1, 3, 4)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(\alpha)} > 0, \quad b_2^{(\alpha)} < 0, \quad b^1 > 0, \quad b^{\alpha_2} = 0, \quad \alpha_1 = \alpha_2, \quad a_2^\alpha < b_2^\alpha$$

For

$$\mu_A(x) = \begin{cases} 0, & x < -4, \ \ 1 < x, \\ x+4, & -4 \le x < -3, \\ -\frac{1}{4}x + \frac{1}{4}, & -3 \le x \le 1, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -1, \ \ 4 < x, \\ \frac{1}{4}x + \frac{1}{4}, & -1 \le x < 3, \\ -x+4, & 3 \le x \le 4, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 4$ and $\alpha = -\frac{a_2^{(\alpha)}}{4} + \frac{1}{4}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 4, -4\alpha + 1]$. Since $\alpha = \frac{b_1^{(\alpha)}}{4} + \frac{1}{4}$ and $\alpha = -b_2^{(\alpha)} + 4$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [4\alpha - 1, -\alpha + 4]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [5\alpha - 5, -5\alpha + 5]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-5, 5]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = 0$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -5, \ \ 5 < x, \\ \frac{1}{5}x + 1, & -5 \le x < 0, \\ -\frac{1}{5}x + 1, & 0 \le x \le 5, \end{cases}$$

i.e., $A(+)B = (-5, 0, 5)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 8, -8\alpha + 2]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-8, 2]^c$ By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -8, \ \ 2 < x, \\ \frac{1}{2}x + 4, & -8 \le x < -6, \\ -\frac{1}{8}x + \frac{1}{4}, & -6 \le x \le 2, \end{cases}$$

i.e., $A(-)B = (-8, -6, 2)$.

(3) Multiplication : (i) $\frac{1}{4} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-\alpha^2 + 8\alpha - 16, -16\alpha^2 + 8\alpha - 1]$, $\mu_{A(\cdot)B}(x) = \frac{1}{4}$ at $x = -\frac{225}{16}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -9$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 4 - \sqrt{x}, & -\frac{225}{16} \le x < -9, \\ \frac{1+\sqrt{-x}}{4}, & -9 \le x < 0. \end{cases}$$

(ii) $0 \le \alpha \le \frac{1}{4}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-\alpha^2+8\alpha-16, 4\alpha^2-17\alpha+4]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-16,4]^c$ and $\mu_{A(\cdot)B}(x) = \frac{1}{4}$ at $x = -\frac{225}{16}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} 4-\sqrt{x}, & -14 \leq x < -\frac{225}{16}, \\ \frac{17-\sqrt{225+16x}}{8}, & 0 \leq x < 4. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{1}{4} < x \leq 1$

Since $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{\alpha-4}{4\alpha-1}, \frac{-4\alpha+1}{-\alpha+4}\right]$ and $\mu_{A(/)B}(x) = 1$ at $x = -1$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{x-4}{4x-1}, & -\infty < x < -1, \\ \frac{4x-1}{x-4}, & -1 \leq x < 0. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 6 : $\alpha_1 < \alpha_2$ and $c < r$

Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \ \ c < x, \\ \frac{1}{a-b}(x+a), & -a \leq x < -b, \\ \frac{1}{c+b}(c-x) & -b \leq x \leq c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \ \ r < x, \\ \frac{1}{q+p}(x+p), & -p \leq x < q, \\ \frac{1}{r-q}(r-x) & q \leq x \leq r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ are the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c-a_2^{(\alpha)}}{c+b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, -\alpha(c+b)+c].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, -\alpha(c+b)+c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{-x+c+r}{c+b+r-q}, & q-b \le x \le c+r. \end{cases}$$

Hence $A(+)B$ is a triangular fuzzy number.

2. Subtraction :
By the above facts, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b)-a+\alpha(r-q)-r, -\alpha(c+b)+c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x) = 0$ on the interval $[-a-r, c+p]^c$ and $\mu_{A(-)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{-x+c+p}{c+b+q+p}, & -b-q \le x \le c+p. \end{cases}$$

Hence $A(-)B$ is a triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a) \cdot (-\alpha(r-q)+r), (-\alpha(c+b)+c) \cdot (\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = \alpha_2(a-b)-a) \cdot (-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c) \cdot (\alpha_2(q+p)-p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_2(a-b)-a) \cdot (-\alpha_2(r-q)+r) \le x < -bq, \\ \frac{bp+cp+be+ce+\sqrt{(-bp-cp-cq-pc)^2-4(bp+cp+be+ce)(cp+x)}}{2(bp+cp+be+ce)}, \\ \qquad -bq \le x < (-\alpha_2(c+b)+c) \cdot (\alpha_2(q+p)-p). \end{cases}$$

(2) $\alpha_1 < \alpha \le \alpha_2$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a) \cdot (-\alpha(r-q)+r), (\alpha(a-b)-a) \cdot (\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a) \cdot (-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a) \cdot (\alpha_1(q+p)-p)$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a) \cdot (-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c) \cdot (\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r) \leq x \\ \qquad < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-ae+bp)^2-4(ap-bp+aq-bq)(ap-x)}}{2(ap-bp+aq-bq)}, \\ (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \leq x \\ \qquad < (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(3) $0 < \alpha \leq \alpha_1$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_2^{(\alpha)}\cdot b_2^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (-\alpha(c+b)+c)\cdot(-\alpha(r-q)+r)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, cr]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (-\alpha_1(c+b)+c)\cdot(-\alpha_1(r-q)+r)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \leq x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{cq-cr+rb+rc-\sqrt{(-cq+cr-rb-rc)^2-4(-bq-cq+bf+cr)(-cr+x)}}{2(-bq-cq+bf+cr)}, \\ \qquad (-\alpha_1(c+b)+c)\cdot(\alpha_1(q+p)-p) \leq x < cr. \end{cases}$$

There is a case like this, or the following. By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_1^{(\alpha)}\cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_1$ at $x = (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r)$ and $x = (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \leq x < (\alpha_1(a-b)-a)\cdot(-\alpha_1(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \leq x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_1 < \alpha \leq 1$

By the above fact, $A_\alpha(/)B_\alpha = \Big(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\Big] = \Big(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(-\alpha(c+b)+c)}{(-\alpha(r-q)+r)}\Big]$ and $\mu_{A(/)B}(\frac{-b}{q}) = 1$. Therefore,

$$\mu_{A(/)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \leq \frac{-b}{q}, \\ \frac{-rx+c}{(q-r)x+c+b}, & \frac{-b}{q} \leq x \leq 0. \end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

Example 3.6. Let $A = (-5, -4, 2)$ and $B = (-1, 1, 4)$ be triangular fuzzy numbers, i.e.,

$$b_1^{(\alpha)} > 0, \quad b_2^{(\alpha)} < 0, \quad b^1 > 0, \quad b^{\alpha_2} = 0, \quad \alpha_1 < \alpha_2, \quad a_2^\alpha < b_2^\alpha$$

For

$$\mu_A(x) = \begin{cases} 0, & x < -5, \ 2 \leq x, \\ x+5, & -5 \leq x < -4, \\ -\frac{1}{6}x + \frac{1}{3}, & -4 \leq x < 2, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -1, \ 4 \leq x, \\ \frac{1}{2}x + \frac{1}{2}, & -1 \leq x < 1, \\ -\frac{1}{3}x + \frac{4}{3}, & 1 \leq x < 4, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)}, \ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, \ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 5$ and $\alpha = -\frac{a_2^{(\alpha)}}{6} + \frac{1}{3}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 5, -6\alpha + 2]$. Since $\alpha = \frac{b_1^{(\alpha)}}{2} + \frac{1}{2}$ and $\alpha = -\frac{b_2^{(\alpha)}}{3} + \frac{4}{3}$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [2\alpha - 1, -3\alpha + 4]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [3\alpha - 6, -9\alpha + 6]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-6, 6]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -3$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -6, \ 6 \leq x, \\ \frac{1}{3}x + 2, & -6 \leq x < -3, \\ -\frac{1}{9}x + \frac{2}{3}, & -3 \leq x < 6, \end{cases}$$

i.e., $A(+)B = (-6, -3, 6)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [4\alpha - 9, -8\alpha + 3]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-9, 3]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -5$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -9, \ \ 3 \le x, \\ \frac{1}{4}x + \frac{9}{4}, & -9 \le x < -5, \\ -\frac{1}{8}x + \frac{3}{8}, & -5 \le x < 3, \end{cases}$$

i.e., $A(-)B = (-9, -5, 3)$.

(3) Multiplication : (i) $\frac{1}{2} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-3\alpha^2 + 19\alpha - 20, -12\alpha^2 + 10\alpha - 2]$, $\mu_{A(\cdot)B}(x) = \frac{1}{2}$ at $x = -\frac{43}{4}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{19-\sqrt{121-12x}}{6}, & -\frac{43}{4} \le x \le -4, \\ \frac{5+\sqrt{1-12x}}{12}, & -4 \le x \le 0. \end{cases}$$

(ii) $\frac{1}{3} \le \alpha \le \frac{1}{2}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] = [-3\alpha^2 + 19\alpha - 20, 2\alpha^2 - 11\alpha + 5]$, $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -14, \frac{14}{9}$ and $\mu_{A(\cdot)B}(x) = \frac{1}{2}$ at $x = -\frac{43}{4}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{19-\sqrt{121-12x}}{6}, & -14 \le x \le -\frac{43}{4}, \\ \frac{11-\sqrt{81+8x}}{4}, & 0 \le x \le \frac{14}{9}. \end{cases}$$

(iii) $\frac{21-\sqrt{249}}{32} \le \alpha \le \frac{1}{3}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}] = [-3\alpha^2 + 19\alpha - 20, 18\alpha^2 - 30\alpha + 8]$, $\mu_{A(\cdot)B}(x) = \frac{21-\sqrt{249}}{32}$ at $x = \frac{-4891-241\sqrt{249}}{512}, \frac{113+51\sqrt{249}}{256}$ and $\mu_{A(\cdot)B}(x) = \frac{1}{3}$ at $x = -14, \frac{14}{9}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{19-\sqrt{121-12x}}{6}, & \frac{-4891-241\sqrt{249}}{512} \le x \le -14, \\ \frac{5-\sqrt{9+2x}}{6}, & \frac{14}{9} \le x \le \frac{113+51\sqrt{249}}{256}. \end{cases}$$

(iv) $0 \le \alpha \le \frac{21-\sqrt{249}}{32}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] = [-3\alpha^2 + 19\alpha - 20, 2\alpha^2 - 11\alpha + 5]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-20, 5]^c$ and $\mu_{A(\cdot)B}(x) = \frac{21-\sqrt{249}}{32}$ at $x = \frac{-4891-241\sqrt{249}}{512}, \frac{113+51\sqrt{249}}{256}$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{19-\sqrt{121-12x}}{6}, & -20 \le x \le \frac{-4891-241\sqrt{249}}{512}, \\ \frac{11-\sqrt{81+8x}}{4}, & \frac{113+51\sqrt{249}}{256} \le x \le 5. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{1}{2} < x \le 1$

Since $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{\alpha-5}{2\alpha-1}, \frac{-6\alpha+2}{-3\alpha+4}\right]$ and $\mu_{A(/)B}(x) = \frac{1}{2}$ on the interval $(-\infty, -\frac{2}{5}]$ and $\mu_{A(/)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{x-5}{2x-1}, & -\infty < x < -4, \\ \frac{4x-2}{3x-6}, & -4 \leq x < -\frac{2}{5}. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 7 :

$a_2^{(0)} > b_1^{(0)}$, $a_1^{(0)} < 0$, $a_2^{(0)} < 0$, $a^{(1)} < 0$
Note that

$$\mu_A(x) = \begin{cases} 0, & x < -a, \ \ c < x, \\ \frac{1}{a-b}(x+a), & -a \leq x < -b, \\ \frac{1}{c-b}(x+c) & -b \leq x < -c, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -p, \ \ r < x, \\ \frac{1}{q+p}(x+p), & -p \leq x < q, \\ \frac{1}{r-q}(r-x) & q \leq x \leq r. \end{cases}$$

We calculate exactly four operations using α-cuts. Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ are the α-cuts of A and B, respectively. Since $\alpha = \frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha = \frac{c+a_2^{(\alpha)}}{c-b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, \alpha(c-b)-c].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above fact, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b) - a + \alpha(q + p) - p, \alpha(c-b) - c - \alpha(r-q) + r]$. Thus $\mu_{A(+)B}(x) = 0$ on interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -c+r, \ \ -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \leq x < q-b, \\ \frac{x+c-r}{c-b-r+p}, & q-b \leq x \leq -c+r. \end{cases}$$

Hence $A(+)B$ is a generalized triangular fuzzy.

2. Subtraction :

By the above fact, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b) - a + \alpha(r - q) - r, \alpha(c-b) - c - \alpha(q+p) + p]$. Thus $\mu_{A(-)B}(x) = 0$ on interval $[-a-r, c+p]^c$ and $\mu_{A(+)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{x+c-p}{c-b-q-p}, & -b-q \le x < -c+p. \end{cases}$$

Hence $A(-)B$ is a generalized triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$
By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(c-b)-c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (\alpha_2(c-b)-c)\cdot(\alpha_2(q+p)-p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r) \le x < -bq, \\ \frac{-bp+cp+bq+cq+\sqrt{(bp-cp-cq-pc)^2-4(-bp+cp-bq+cq)(cp-x)}}{2(-bp+cp-bq+cq)}, \\ \qquad -bq \le x < (\alpha_2(c-b)-c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(2) $0 \le \alpha \le \alpha_2$
By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar \le x < (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r), \\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p) \le x < ap. \end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_2 < \alpha \le 1$

By the above fact, $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(\alpha(c-b)-c)}{(-\alpha(r-q)+r)}\right]$ and $\mu_{A(/)B}(x) = 1$ at $x = \frac{-b}{q}$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{px-a}{(q+p)x-(a-b)}, & -\infty < x \leq \frac{-b}{q}, \\ \frac{rx-c}{(r-q)x-c+b}, & \frac{-b}{q} \leq x \leq 0. \end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

EXAMPLE 3.7. Let $A = (-4, -3, -1)$ and $B = (-2, 1, 2)$ be triangular fuzzy numbers, i.e.

For

$$\mu_A(x) = \begin{cases} 0, & x < -4, \ \ -1 \leq x, \\ x + 4, & -4 \leq x < -3, \\ -\frac{1}{6}x + \frac{1}{3}, & -3 \leq x < -1, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & x < -2, \ \ 2 \leq x, \\ \frac{1}{3}x + \frac{2}{3}, & -2 \leq x < 1, \\ -x + 2, & 1 \leq x < 2, \end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)},\ b_2^{(\alpha)}]$. Since $\alpha = a_1^{(\alpha)} + 4$ and $\alpha = -\frac{a_2^{(\alpha)}}{2} - \frac{1}{2}$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 4, -2\alpha - 1]$. Since $\alpha = \frac{b_1^{(\alpha)}}{3} + \frac{2}{3}$ and $\alpha = -b_2^{(\alpha)} + 2$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [3\alpha - 2, -\alpha + 2]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [4\alpha - 6, -3\alpha + 1]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-6, 1]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -2$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -6, \ \ 1 \leq x, \\ \frac{1}{4}x + \frac{3}{2}, & -6 \leq x < -2, \\ -\frac{1}{3}x + \frac{1}{3}, & -2 \leq x < 1, \end{cases}$$

i.e., $A(+)B = (-6, -2, 1)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 6, -5\alpha + 1]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-6, 1]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -6, \ \ 1 \leq x, \\ \frac{1}{2}x + 3, & -6 \leq x < -4, \\ -\frac{1}{5}x + \frac{1}{5}, & -4 \leq x < 1, \end{cases}$$

i.e., $A(-)B=(-6,-4,1)$.

(3) Multiplication : (i) $\frac{2}{3}\le\alpha\le 1$

Since $A_\alpha(\cdot)B_\alpha=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_2^{(\alpha)}\cdot b_1^{(\alpha)}]=[-\alpha^2+6\alpha-8,-6\alpha^2+\alpha+2]$, $\mu_{A(\cdot)B}(x)=\frac{2}{3}$ at $x=-\frac{40}{9},0$ and $\mu_{A(\cdot)B}(x)=1$ at $x=-3$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}3-\sqrt{1-x}, & -\frac{40}{9}\le x<-3,\\ \frac{1+\sqrt{49-24x}}{12}, & -3\le x<0.\end{cases}$$

(ii) $0\le\alpha\le\frac{2}{3}$

Since $A_\alpha(\cdot)B_\alpha=[a_1^{(\alpha)}\cdot b_2^{(\alpha)},a_1^{(\alpha)}\cdot b_1^{(\alpha)}]=[-\alpha^2+6\alpha-8,3\alpha^2-14\alpha+8]$, $\mu_{A(\cdot)B}(x)=0$ on the interval $[-8,8]^c$ and $\mu_{A(\cdot)B}(x)=\frac{2}{3}$ at $x=-\frac{40}{9},0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x)=\begin{cases}3-\sqrt{1-x}, & -8\le x<-\frac{40}{9},\\ \frac{7-\sqrt{3x+25}}{2}, & 0\le x<8.\end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{2}{3}<x\le 1$

Since $A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}},\frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left(\frac{\alpha-4}{3\alpha-2},\frac{-2\alpha-1}{-\alpha+2}\right]$ and $\mu_{A(/)B}(x)=\frac{2}{3}$ on the interval $(-\infty,-\frac{7}{4}]$ and $\mu_{A(/)B}(x)=1$ at $x=-3$. By the routine calculation, we have

$$\mu_{A(/)B}(x)=\begin{cases}\frac{2x-4}{3x-1}, & -\infty<x<-3,\\ \frac{2x+1}{x-2}, & -3\le x<-\frac{7}{4}.\end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

CASE 8 :

$a_2^{(0)}<b_1^{(0)},\quad a_1^{(0)}<0\ ,\quad a_2^{(0)}<0,\quad a^{(1)}<0$

Note that

For

$$\mu_A(x)=\begin{cases}0, & x<-a,\ \ c<x,\\ \frac{1}{a-b}(x+a), & -a\le x<-b,\\ \frac{1}{c-b}(x+c) & -b\le x<-c,\end{cases}$$

and

$$\mu_B(x)=\begin{cases}0, & x<-p,\ \ r<x,\\ \frac{1}{q+p}(x+p), & -p\le x<q,\\ \frac{1}{r-q}(r-x) & q\le x<r,\end{cases}$$

we calculate exactly four operations using α-cuts. Let $A_\alpha=[a_1^{(\alpha)},a_2^{(\alpha)}]$ and $B_\alpha=[b_1^{(\alpha)},b_2^{(\alpha)}]$ are the α-cuts of A and B, respectively. Since $\alpha=\frac{a_1^{(\alpha)}+a}{a-b}$ and $\alpha=\frac{c+a_2^{(\alpha)}}{c-b}$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha(a-b)-a, \alpha(c-b)-c].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\alpha(q+p)-p, -\alpha(r-q)+r].$$

1. Addition :

By the above fact, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [\alpha(a-b)-a+\alpha(q+p)-p, \alpha(c-b)-c-\alpha(r-q)+r]$. Thus $\mu_{A(+)B}(x) = 0$ on interval $[-a-p, c+r]^c$ and $\mu_{A(+)B}(q-b) = 1$. Therefore,

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -c+r, \quad -a-p < x, \\ \frac{x+a+p}{a-b+q+p}, & -a-p \le x < q-b, \\ \frac{x+c-r}{c-b-r+p}, & q-b \le x \le -c+r. \end{cases}$$

Hence $A(+)B$ is a generalized triangular fuzzy.

2. Subtraction :

By the above fact, $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [\alpha(a-b)-a+\alpha(r-q)-r, \alpha(c-b)-c-\alpha(q+p)+p]$. Thus $\mu_{A(-)B}(x) = 0$ on interval $[-a-r, c+p]^c$ and $\mu_{A(+)B}(-b-q) = 1$. Therefore,

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -c+p, \quad -a-r < x, \\ \frac{x+a+r}{a-b+r-q}, & -a-r \le x < -b-q, \\ \frac{x+c-p}{c-b-q-p}, & -b-q \le x \le -c+p. \end{cases}$$

Hence $A(-)B$ is a generalized triangular fuzzy number.

3. Multiplication : (1) $\alpha_2 < \alpha \le 1$

By the above fact,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] \\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(c-b)-c)\cdot(\alpha(q+p)-p)]. \end{aligned}$$

Thus $\mu_{A(\cdot)B}(x) = \alpha_2$ at $x = (\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x = (\alpha_2(c-b)-c)\cdot(\alpha_2(q+p)-p)$, $\mu_{A(\cdot)B}(-bp) = 1$. Therefore,

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad \alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r) \le x < -bq, \\ \frac{-bp+cp+bq+cq+\sqrt{(bp-cp-cq-pc)^2-4(-bp+cp-bq+cq)(cp-x)}}{2(-bp+cp-bq+cq)}, \\ \qquad -bq \le x < (\alpha_2(c-b)-c)\cdot(\alpha_2(q+p)-p). \end{cases}$$

(2) $0 \le \alpha \le \alpha_2$

By the above facts,

$$\begin{aligned}A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}\cdot b_2^{(\alpha)}, a_1^{(\alpha)}\cdot b_1^{(\alpha)}]\\ &= [(\alpha(a-b)-a)\cdot(-\alpha(r-q)+r), (\alpha(a-b)-a)\cdot(\alpha(q+p)-p)].\end{aligned}$$

Thus $\mu_{A(\cdot)B}(x)=0$ on the interval $[-ar, ap]^c$ and $\mu_{A(\cdot)B}(x)=\alpha_2$ at $x=(\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r)$ and $x=(-\alpha_2(c+b)+c)\cdot(\alpha_2(q+p)-p)$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{aq-2ar+rb-\sqrt{(-aq-2ar-bp)^2-4(-aq+bq+ar-br)(ar+x)}}{2(-aq+bq-ar-br)}, \\ \qquad -ar\le x<(\alpha_2(a-b)-a)\cdot(-\alpha_2(r-q)+r),\\ \frac{2ap+aq-bp-\sqrt{(-2ap-aq+bp)^2-4(ap-bp+aq-bq)(ad-x)}}{2(ap-bp+aq-bq)}, \\ \qquad (\alpha_1(a-b)-a)\cdot(\alpha_1(q+p)-p)\le x<ap.\end{cases}$$

Hence $A(\cdot)B$ is a fuzzy number, but need not to be a generalized triangular fuzzy set.

4. Division : (1) $\alpha_2<\alpha\le 1$

By the above fact, $A_\alpha(/)B_\alpha=\left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right]=\left(\frac{(\alpha(a-b)-a)}{(\alpha(q+p)-p)}, \frac{(\alpha(c-b)-c)}{(-\alpha(r-q)+r)}\right]$ and $\mu_{A(/)B}(x)=1$ at $x=\frac{-b}{q}$. Therefore,

$$\mu_{A(\cdot)B}(x)=\begin{cases}\frac{px-a}{(q+p)x-(a-b)}, & -\infty<x\le\frac{-b}{q},\\ \frac{rx-c}{(r-q)x-c+b}, & \frac{-b}{q}\le x\le 0.\end{cases}$$

Hence $A(/)B$ becomes a fuzzy set on $\mathbb{R}$.

EXAMPLE 3.8. Let $A=(-5,-4,-3)$ and $B=(-2,1,2)$ be triangular fuzzy numbers, i.e.,

For

$$\mu_A(x)=\begin{cases}0, & x<-5,\ -3\le x,\\ x+5, & -5\le x<-4,\\ -x-3, & -4\le x<-3,\end{cases}$$

and

$$\mu_B(x)=\begin{cases}0, & x<-2,\ 2\le x,\\ \frac{1}{3}x+\frac{2}{3}, & -2\le x<1,\\ -x+2, & 1\le x<2,\end{cases}$$

we calculate exactly the above four operations using $\alpha-$ cuts. Let A_α and B_α be the α-cuts of A and B, respectively. Let $A_\alpha=[a_1^{(\alpha)},\ a_2^{(\alpha)}]$ and $B_\alpha=[b_1^{(\alpha)},\ b_2^{(\alpha)}]$.

Since $\alpha = a_1^{(\alpha)} + 5$ and $\alpha = -a_2^{(\alpha)} - 3$, we have $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [\alpha - 5, -\alpha - 3]$.
Since $\alpha = \frac{b_1^{(\alpha)}}{3} + \frac{2}{3}$ and $\alpha = -b_2^{(\alpha)} + 2$, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [3\alpha - 2, -\alpha + 2]$.

(1) Addition :

By the above facts, $A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] = [4\alpha - 7, -2\alpha - 1]$. Thus $\mu_{A(+)B}(x) = 0$ on the interval $[-7, -1]^c$ and $\mu_{A(+)B}(x) = 1$ at $x = -3$. By the routine calculation, we have

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < -7, \quad -1 \le x, \\ \frac{1}{4}x + \frac{7}{4}, & -7 \le x < -3, \\ -\frac{1}{2}x - \frac{1}{2}, & -3 \le x < -1, \end{cases}$$

i.e., $A(+)B = (-7, -3, -1)$.

(2) Subtraction :

Since $A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] = [2\alpha - 7, -4\alpha - 1]$, we have $\mu_{A(-)B}(x) = 0$ on the interval $[-7, -1]^c$ and $\mu_{A(-)B}(x) = 1$ at $x = -5$. By the routine calculation, we have

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < -7, \quad -1 \le x, \\ \frac{1}{2}x + \frac{7}{2}, & -7 \le x < -5, \\ -\frac{1}{4}x - \frac{1}{4}, & -5 \le x < -1, \end{cases}$$

i.e., $A(-)B = (-7, -5, -1)$.

(3) Multiplication : (i) $\frac{2}{3} \le \alpha \le 1$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_2^{(\alpha)} \cdot b_1^{(\alpha)}] = [-\alpha^2 + 7\alpha - 10, -3\alpha^2 - 7\alpha + 6]$, $\mu_{A(\cdot)B}(x) = \frac{2}{3}$ at $x = -\frac{52}{9}, 0$ and $\mu_{A(\cdot)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 4x}}{2}, & -\frac{52}{9} \le x < -4, \\ \frac{-7 + \sqrt{121 - 12x}}{6}, & -4 \le x < 0. \end{cases}$$

(ii) $0 \le \alpha \le \frac{2}{3}$

Since $A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)} \cdot b_2^{(\alpha)}, a_1^{(\alpha)} \cdot b_1^{(\alpha)}] = [-\alpha^2 + 7\alpha - 10, 4\alpha^2 - 17\alpha + 10]$, $\mu_{A(\cdot)B}(x) = 0$ on the interval $[-10, 10]^c$ and $\mu_{A(\cdot)B}(x) = \frac{2}{3}$ at $x = -\frac{52}{9}, 0$. By the routine calculation, we have

$$\mu_{A(\cdot)B}(x) = \begin{cases} \frac{7 - \sqrt{9 - 4x}}{2}, & -10 \le x < -\frac{52}{9}, \\ \frac{17 - \sqrt{44x - 151}}{8}, & 0 \le x < 10. \end{cases}$$

Thus $A(\cdot)B$ is not a triangular fuzzy number.

(4) Division : (i) $\frac{2}{3} < x \le 1$

Since $A_\alpha(/)B_\alpha = \left(\frac{a_1^{(\alpha)}}{b_1^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_2^{(\alpha)}}\right] = \left(\frac{\alpha - 5}{3\alpha - 2}, \frac{-\alpha - 3}{-\alpha + 2}\right]$ and $\mu_{A(/)B}(x) = \frac{2}{3}$ on the interval $(-\infty, -\frac{11}{4}]$ and $\mu_{A(/)B}(x) = 1$ at $x = -4$. By the routine calculation, we have

$$\mu_{A(/)B}(x) = \begin{cases} \frac{2x-5}{3x-1}, & -\infty < x < -4, \\ \frac{2x+3}{x-1}, & -4 \le x < -\frac{11}{4}. \end{cases}$$

Thus $A(/)B$ is not a triangular fuzzy number.

Generally, for two triangular fuzzy numbers A and B, $A(+)B$ and $A(-)B$ always becomes triangular fuzzy numbers. But $A(\cdot)B$ and $A(/)B$ may not be triangular fuzzy numbers.

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Hyun Kim
Department of Mathematics
Jeju National University
E-mail: badamul0313@naver.com

<국문 초록>

두 개의 양수가 아닌 삼각퍼지수에 대한 Zadeh의 확장연산

전통적인 논리는 참 또는 거짓입니다. 원소가 어떤 집합에 속하는가? 또는 속하지 않는가? 하는 이분법적인 논리는 실제로 우리 삶의 일부분만을 나타냅니다. 불확실하고 복잡한 삶의 특성을 잘 표현하기 위해서는 보다 확장된 논리 개념과 명제 개념이 필요합니다. 그래서 이 문제에 대한 해결책으로, Zadeh는 1960년대 초에 퍼지 집합을 도입했습니다. 퍼지 집합은 원소가 특정집합에 속하거나 속하지 않는 두 가지 경우에만 국한되지 않습니다.

퍼지 집합이론에서는 두 가지 퍼지 집합사이의 다양한 유형의 연산이 정의되고 연구되었습니다. 특히 두 개의 삼각퍼지수 사이의 확장대수 연산의 많은 결과가 잘 알려져 있습니다. 이 논문에서는 두 개의 양이 아닌 삼각퍼지수에 대한 Zadeh의 확장연산을 계산합니다.