On Decomposition of Certain Topological Spaces

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Summary

In this paper, some properties of a mapping f in certain topological spaces are defined and proved. Using these properties, we prove that if f:S → T is a peripherally continuous mapping, then the null collection is an upper semi-continuous decomposition of the domain space.

1. Introduction

Even though the space of continuous function has been intensively studied in functional analysis, semi-continuous functions could be used to introduce the important properties of continuity. This paper is concerned with the relationship between certain non-continuous functions and decomposition of the domain space into upper semi-continuous collections. Related to this is the factorization of functions and the properties possessed by the factors.

2. Preipherally Continuous Mapping

This section presents definitions and properties of the fundamental concepts of a preipherally continuous mapping and a connectivity map of a mapping and locally peripherally connectedness of a space.

Definition (2.1): A mapping f : S → T is peripherally continuous if for every point p in S and for every pair of open sets U and V containing p and f(p), respectively, there is an open set N⊂U containing p such that f(F(N))⊂V, where F(N) denotes the boundary of N.

Definition (2.2): The mapping f is a connectivity map if for every connected set A in S the set g(A) is connected, where g : S → S x T is the graph map induced by f and defined by g(p)=(p,f(p)).

Remark: If f is peripherally continuous, then the graph map g is also peripherally continuous and conversely.

Definition (2.3): A space S is locally peripherally connected if for every point p in S and every open set U containing p, there is an open set V⊂U and containing p such that F(V) is connected.

A useful characterization of an upper semi-continuous collection in a compact metric space S is as follows; "A necessary and sufficient condition that a collection G of closed sets be upper semi-continuous is that for any $\{g_n\}$ of elements of G with $g \cap (\liminf g_n) \neq \phi$, where g∈G, then lim sup $g_n \subset g$."

The theorem is proved by G.T. Whyburn, 1963.

Throughout this paper, unless otherwise stated, S will denote a locally peripherally connected, compact, separable metric space and T a regular Hausdorff space. The definition of locality, compactness and other definition for separation axioms are followed J.L. Kelley's, 1963.

Theorm (2.4): If f is a peripherally continuous mapping of the locally peripherally connected space S into the space T, then for every point p in S and every pair of open sets U and V containing p and f(p), respectively, there is an open connected set $N \subset U$ and containing p such that F(N) is connected and $f(F(N)) \subset V$.

Proof: Using the result of J. Stallings, 1959, we obtain the theorem.

3. The Properties of f^{-1} (C)

If f is a continuous function from a space S into a space T and C is a closed subset of T, the set f^{-1} (C) is closed. Furthermore P.E. Long, 1961 showed following; "For a connectivity map or a peripherally continuous function the components of f^{-1} (C) are closed."

The following two theorems and the resulting corollaries give some further information concerning f^{-1} (C).

Theorem (3.1): If $f : S \to T$ is a connectivity map, C is a closed subset of T and S is semi-locally connected, then the components of f^{-1} (C) form a semi-closed collection.

Proof: Since f is a connectivity map and C is closed, the components of f^{-1} (C) are closed (O.H. Hamilton, 1959), let $\{g_n\}$ be a convergent sequence of components of f^{-1} (C) with a non-empty limiting set L. Since S is compact, $\bar{\cup}\, g_n$ is compact and hence L is connected (J. Stallings, 1959).

Suppose L is non-degenerate and that $L \cap (S - f^{-1}(C)) \neq \phi$. Let x be a point of L such that x is not in f^{-1} (C) and let y be a point of L distinct from x. If no such point exists, then L = { x } and f^{-1} (C) is semi-closed. Then there is a sequence of point $\{y_n\}$ of L converging to y. Since S is semi-locally connected, there is an open set U containing x such that y_n is not in U, n = 1, 2,....., and S – U has a finite number of components K_i, i = 1, 2,....., j. Since there are only a finite number of the K_i, some K_i must intersect infinitely many g_n since y_n is in S – U for all n. Denote these by g_n'. Then $E = K_i \cup (\cup g_n') \cup \{x\}$ is a connected subset of S and f a connectivity map implies that the graph g(E) is connected.

Now $U \cap K_i = \phi$ and $f(\cup g_n') \cap (T-C) = \phi$ since $f(\cup g_n') \subset C$. But x is in U and f(x) is in T-C since x is not in f^{-1} (C). Thus U x (T-C) is an open set in S x T containing only the point g(x) of g(E). This contradicts g(E) being connected. Therefore either L is contained in f^{-1} (C) or L is a single point. Thus the components of f^{-1} (C) form a semi-closed collection.

Theorem (3.2): If $f : S \to T$ is a preipherally continuous mapping and C is a closed subset of T, then the components of f^{-1} (C) form a semi-closed collection.

Proof: The components of f^{-1} (C) are closed by the result of P.E. Long, 1961. Let $\{g_n\}$ be a convergent sequence of components of f^{-1} (C) and let $\lim g_n = L$.

Suppose $L \cap (S - f^{-1}(C)) \neq \phi$ and let $a \in L \cap (S - f^{-1}(C))$. Let b be any other point of L. If no such point exists, then L = {a} and f^{-1} (C) is semi-closed. Since $\{g_n\}$ is a sequence of connected sets and $\bar{\cup}\, g_n$ is compact, L is connected (G.T. Whyburn, 1963). Since L contains the two distinct point a and b, L is non-degenerate and hence there is $\varepsilon > 0$ such that diameter $g_n \geq \varepsilon$ for every n.

Let $\{U_n\}$ and $\{V_n\}$ be a sequences of open sets closing down on a and f(a), respectively, such that diameter $U_n < \varepsilon$, $F(U_n)$ is connected, and $f(F(U_n)) \subset V_n$, for every n. Since diameter $g_n \geq \varepsilon$, diameter $U_n < \varepsilon$ and $F(U_n)$ and g_n are connected, it follows that $F(U_n) \cap g_n \neq \phi$. Let a_n be a point of $F(U_n) \cap g_n$. Since the sequences $\{U_n\}$ and $\{V_n\}$ are closing down on a and f(a), respectively, $a_n \to a$ and $f(a_n) \to f(a)$. But $a_n \in g_n$ implies that $f(a_n) \in C$ and $a \in L \cap (S - f^{-1}(C))$ implies $f(a) \in C$. Thus, f(a) is a limit point of C not in C contradicting that C is closed. Therefore either $L \subset f^{-1}$ (C) or L is a single point, and the components of f^{-1} (C) form a semi-closed collection.

4. Decomposition of S

We have now the following results.

Theorem (4.1): If $f : S \to T$ is a connectivity map and S′ is a null collection, the S′ is upper semi-continuous.

Proof: Since each point $y \in T$ is closed set and f is a connectivity map, the components of $f^{-1}(y)$ are closed. The S′ is a null collection of disjoint closed sets and is therefore upper semi-continuous (G.T. Whyburn, 1963).

Theorem (4.2): If $f : S \rightarrow T$ is a preipherally continuous mapping, then S′ is an upper semi-continuous decomposition of S.

Proof: The elements of S′ are closed since y is closed in T and f is preipherally continous (P.E. Long, 1961), since S is compact, S′ is a collection of disjoint compact cointinua filling up S. Here, using the result of R.L. Moore, 1962 and G.T. Whyburn, 1963, we obtain the result.

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〈国文抄録〉

本 論文에서는 位相空間 內에서 사상 f 가 갖는 特殊한 性質을 定義하고 이들로 부터 null collection이 f의 定義域의 上半連續分割이 됨을 보인다.