A note on the Topological R_0-regular spaces

Bang Eun-sook

R_0-regular 位相空間에 關하여

方 銀 淑

Summary

The category R_0-*Reg* of topological R_0-regular spaces and continuous maps is a bireflective subcategory of the cateogry *Top* of topological spaces and continuous maps. Moreover R_0-*Reg* is productive and a topological category.

I. INTRODUCTION

In 1975, C.Y. Kim & Park [1] have introduced the concept of R-proximity which has been some generalization of the concept of Efremovič's proximity. It is shown that the R-proximity δ induces a topology $\tau(\delta)$ in X and this induced topology is R_0-regular and also shown that every topological R_0-regular space (X,τ) admits a compatible R-proximity δ on X such that $\tau(\delta)=\tau$.

In particular, all nearness space (X,ξ) are symmetric in the sense that $x\in cl_\xi\{y\}$ impies $y\in cl_\xi\{x\}$ (see[3]). Thus it is well known that the category of topological R_0-spaces and continuous maps is isomorphic with the category of topological nearness space and nearness preserving maps. (see [3]).

The purpose of the present note is to characterize some properties in the topological R_0-regular space using the categorical terminologies.

In this note, the category of topological spaces and continuous maps will be denoted by *Top* and the full subcategory of *Top* whose objects are topological R_0-regular spaces denoted by R_0-*Reg*.

2. PRELIMINARIES

2.1. DEFINITION. A topological space (X,ξ) is R_0 iff either of the following condition is satisfied:

(1) $x\in cl\{y\}$ iff $y\in cl\{x\}$

(2) $x\in G\in\xi$ implies $x\in cl\{x\}\subset G$, whose cl denote the closure operator on X.

2.2. DEFINITION. Leg $G:\mathbf{A}\to\mathbf{B}$ be a functor and B an object of **B**. A pair (u,A) with $A\in\mathbf{A}$ and $u:B\to GA$ is called a universal map for B with respect to G (or a G-universal for B) provided that for each $A'\in\mathbf{A}$ and each $f:B\to GA'$ there exists a unique **A**-morphism $f':A\to A'$ with $(Gf')u=f$

2.3. DEFINITION. A functor $F:\mathbf{A}\to\mathbf{B}$ is said to be embedding provided that $F:Mor(\mathbf{A})\to Mor(\mathbf{B})$ from morphisms of **A** to morphisms of **B** is an injective function.

2.4. DEFINITION. Let **A** be a subcategory of **B** with the embedding functor $F:\mathbf{A}\to\mathbf{B}$ an F-universal map for a **B**-object B is called an **A**-reflection of B and **A** is called a reflective subcategory of **B** if there exists an A-reflection for each **B**-objects.

2.5. DEFINITION. Let **A** be a concrete category and $(Y_i, \xi_i)_{i\in I}$ a family of objects in **A** indexed by a class I, and let X be a set and $(f_i: X \to Y_i)_{i\in I}$ a source of maps indexed by I. An **A**-structure ξ on X is called initial with respect to $(X,(f_i)_{i\in I}, (Y_i,\xi_i)_{i\in I})$ if the following conditions are satisfied:

(1) For each $i\in I$, $f_i:(X,\xi) \to (Y_i,\xi_i)$ is an **A**-morphism

(2) If (Z,ζ) is an **A**-object and $g:Z \to X$ is a map such that for each $i\in I$ the map $f_i g:(Z,\zeta) \to (Y_i,\xi_i)$ is an **A**-morphsm, then $g:(Z,\zeta) \to (X,\xi)$ is an **A**-morphism. In this case the source $(f_i:(X,\xi) \to (Y_i,\xi_i))_{i\in I}$ is also called initial.

2.6. REMARK. In the category *Top*, the initial structure is precisely the initial topolpgical structure in the sense of Bourbaki [4].

2.7. DEFINITION. A concrete category **A** is called topological if for each set X, for any family $(Y_i,\xi_i)_{i\in I}$ of **A**-objects, and for any family $(f_i:X \to Y_i)_{i\in I}$ of maps there exists an **A**-structure on X which is initial with respect to $(X,(f_i)_{i\in I}, (Y_i,\xi_i)_{i\in I})$.

2.8. REMARK. The category *Top* is topological.

3. SOME PROPERTIES OF R_o-*Reg*

The following two theorems are useful in this section.

3.1. THEOREM. (see [3] & [5])

If **A** is a topological category and **B** is a full isomorphism closed subcategory of **A**, then the followings are equivalent:

(1) **B** is a bireflective subcategory of **A**.

(2) **B** is closed under the formation of initial source in **A**.

3.2. THEOREM (see [5]). If **A** is a topological category and **B** is a full isomorphism closed bireflective subcategory of **A**, then **B** is also a topological category. Moreover if a source $(f_i:X \to X_i)$ is initial in **A** and for each $i\in I$, $X_i\in$ **B**, then it is initial in **B**.

3.3. THEOREM. *Ro-Reg is closed under the formation of* initial source in *Top*.

PROOF. Suppose $(f_i:(X,\xi) \to (Y_i,\xi_i))_{i\in I}$ is an initial soure in *Top* such that for each $i\in I$, (Y_i, ξ_i) is an initial object of R_o-*Reg*. Let (X,ζ) be an object of R_o-*Reg* which is weaker then ξ, and let $id_X:(X,\zeta) \to (X,\xi)$ be an identity map such that for each $i\in I$, the map $f_i id_X:(X,\zeta) \to (Y_i,\xi_i)$ is continuous.

Then $f_i id_X$ is continuous for each $i\in I$ iff id_X is continuous.

On the other hand, $id_X^{-1}:(X,\xi) \to (X,\zeta)$ is also continuous. So that ξ is the unique initial topology on X with respect to $(f_i)_{i\in I}$. Hence $(X,\xi)\in R_o$-*Reg*.

3.4. PROPOSITION. R_o-*Reg* is a bireflective subcategory of *Top*.

PROOF. It is immediate from 3.1. In particular, the identity map $id_X:(X,\zeta) \to (X,\xi)$ is the R_o-*Reg* reflection of (X,ζ) in the proof of 3.3.

3.3. COROLLARY. R_o-*Reg* is a topological category.

PROOF. It is obvious from 3.2.

3.6. COROLLARY. The product of topological R_o-regular spaces is a topological R_o-regular space. (i.e. R_o-*Reg* is productive).

PROOF. See [2], [4] & [5].

REFERENCES

[1] C.Y. Kim & Chung & Park, 1976. On the R-proximity spaces, Yonsei Nonchong, Vol. 13.

[2] H. Herrlich & G.E. Strecker, Category Theory. (Allyn & Bacon, Boston, 1973.).

[3] H. Herrlich, Topological Structures, 1974. Math. Centre Tract 52.

[4] N. Bourbaki, General Topology I, (Addison-Wesley, London, 1966).

[5] S.S. Hong & Y.H. Hong & P.U. Park. 1979. Algebras in Cartesian Closed Topological Categories, Lect Note Series 1. Yonsei University.

國 文 抄 錄

Ro-regular 위상공간과 연속함수들 전체의 category Ro-Reg는 위상공간과 연속함수들 전체의 category TOP의 bireflective subcategory가 됨을 보였다. 또한 이 Ro-Reg는 可構的이며 位相的 category가 된다.