# On some Properties of the Nonholonomic Components in $V_n$

By

Koh, Aeja

Department of Mathematics
Graduate School of Education
Cheju National University



Supervised By

Assistant Prof. Han, Chulsoon

June, 1981

# On some Properties of the Nonholonomic Components in $V_n$



이를 教育学碩士学位 論文으로 提出함

濟州大学 教育大学院 数学教育專攻

提 出 者 高 愛 子

指導教授 韓 哲 淳

1981 年 6月 日

# 高愛子의 碩士學位 論文을 認准함

濟州大学 教育大学院

主 審 柳 根 植

副 審

副 審



1981年 6月 日

# 감 사 의 말

본 논문을 작성함에 있어서 처음부터 끝까지 애써주신 한철순 교수님께 충심으로 감사를 드립니다.

아울러 그동안 많은 지도와 편달을 하여 주신 수학과 여러 교수님들께도 심심한 감사를 드립니다.

1981년 월 일

고 애 자

# CONTENTS

ABSTRACT ( KOREAN )

1. INTRODUCTION ........ 1

2. PRELIMINARY RESULTS ........ 2

3. SOME RESULTS ........ 3

4. REFERENCES ........ 5

ABSTRACT ( ENGLISH )

# 국 문 초 록

RIEMANNIAN 공간 Vn에서의 NONHOLONOMIC COMPONENTS의 몇가지 성질에 관하여

제주대학교육대학원

수학교육 전공

고 애 자

본 논문에서는, 첫째로 RIEMANNIAN 공간 Vn에서의 NONHOLONOMIC FRAMES와 ORTHOGONAL NONHOLONOMIC FRAMES의 일반적인 구조를 소개하고, 둘째로 ORTHOGONAL NONHOLONOMIC FRAMES의 특수한 성질을 얻은 다음,

마지막으로 이 개념과 성질을 써서, RIEMANNIAN 기하학에서 이미 잘 알려진 몇가지 결과를 보다 더 새롭고 쉬운 방법으로 증명한다.

## 1. INTRODUCTION.

The concept of the nonholonomic frames are introduced by V. Hlavaty 1957 with a set of 4 linearly independent basic null vectors.

In our previous paper [1], [2] introduced the general nonholonomic frames and orthogonal nonholonomic frames to an n-dimensional Riemannian space $V_n$ and constructed the characteristic orthogonal nonholonomic frames of $V_n$ determined by a symmetric tensor $a_{\lambda\mu}$.

The purpose of the present paper, using the definition and properties, prove some well-known results of Riemannian geometry in a new method.

## 2. PRELIMINARY RESULTS

Let $V_n$ be a n-dimensional Riemannian space referred to a real coordinate system $x^\nu$ and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

(2.1) $$h \overset{\text{def}}{=} \mathrm{Det}((h_{\lambda\mu})) \neq 0.$$

According to(2.1) there is a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

(2.2) $$h_{\lambda\mu}\, h^{\lambda\nu} \overset{\text{def}}{=} \delta^\nu_\mu.$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and lowing indices of tensor quantities in $V_n$ in the usual manner.

If $\underset{i}{e}{}^\nu$, (i=1,2,....,n),are a set of n linearly independent unit vectors,then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$, (i=1,2,....,n), satisfying

(2.3) $$\underset{i}{e}{}^\nu\, \overset{i}{e}_\lambda = \delta^\nu_\lambda, \quad \underset{j}{e}{}^\lambda\, \overset{i}{e}_\lambda = \delta^i_j.$$

With the vectors $\underset{i}{e}{}^\nu$ and $\overset{i}{e}_\lambda$ a nonholonomic frame of $V_n$ is defined in the following way;

If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(2.4)a* $$T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} T^{\nu\cdots}_{\lambda\cdots}\, \overset{i}{e}_\nu\, \underset{j}{e}{}^\lambda \cdots$$

From (2.3) and(2.4)a,

(2.4)b $$T^{\nu\cdots}_{\lambda\cdots} = T^{i\cdots}_{j\cdots}\, \underset{i}{e}{}^\nu\, \overset{j}{e}_\lambda \cdots$$

(*) Throughout the present paper, Greek indices are used for the holonomic componenents of a tensor, while Roman indices are used for the nonholonomic components of a tensor. Both indices take the values 1,2,....n. and follow the summation convention

The nonholonomic frame in $V_n$ constructed by the unit vectors $\underset{i}{e}^{\nu}$, (i=1,2, ..., n), tangent to the n congruences of an orthogonalennuple, will be termed an orthogonal nonholonomic frame of $V_n$.

with respect to an orthogonal nonholonomic frame of $V_n$, we have

(2.5)

$$h_{ij} = \delta_{ij}, \quad h^{ij} = \delta^{ij},$$
$$\underset{i}{e}^{\nu} = \overset{i}{e}^{\nu}, \quad \overset{j}{e}_{\lambda} = \underset{j}{e}_{\lambda}.$$

The tensor $h_{\lambda\mu}$, $h^{\lambda\mu}$ and $\delta^{\nu}_{\lambda}$ may be expressed in terms of $\underset{i}{e}$, as follows;

(2.6)

$$h_{\lambda\mu} = \sum_i \underset{i}{e}_{\lambda} \underset{i}{e}_{\mu}$$
$$h^{\lambda\mu} = \sum_i \underset{i}{e}^{\lambda} \underset{i}{e}^{\mu}$$
$$\delta^{\nu}_{\lambda} = \sum_i \underset{i}{e}_{\lambda} \underset{i}{e}^{\nu}$$



## 3. SOME RESULTS.

In this section, we derive the results concerning the n-dimensional Riemannian space $V_n$ employing the newly established nonholonomic frame in the preceeding section.

Consider a symmetric covariant tensors, whose determinant

(3.1) $$a \overset{def}{=} \mathrm{Det}((a_{\lambda\mu})) \neq 0 .$$

It is well-known that the quantity a defined by

(3.2) $$a^{\lambda\nu} \overset{def}{=} \frac{\text{cofactor of } a_{\lambda\nu} \text{ in } a}{a}$$

is a symmetric contravariant tensor satisfying

(3.3) $$a_{\lambda\mu} a^{\lambda\nu} = \delta^{\nu}_{\mu}.$$

THEOREM.(3.1). If the nonholonomic covariant tensor $a_{ij} = 0$ $(i \neq j)$ then

(3.1)a $\quad a^{ij} = 0$ $(i \neq j)$, and $a^{ii} = \frac{1}{a_{ii}}$ $(a_{ii} \neq 0)$

PROOF. Since $a_{\lambda\mu} = 0$ then $a^{\lambda\mu} = 0$, the first relation of(3.1)a follows from(2.4)a, and the second relation may be derived as

$$a_{ij}\ a^{ik} = a_{\lambda\mu} \underset{i}{e}^{\lambda} \underset{j}{e}^{\mu} a^{\lambda\nu} \overset{i}{e}_{\lambda} \overset{k}{e}_{\nu} = \delta^{k}_{j}\ .$$

THEOREM.(3.2). If the tensors $a_{\lambda\mu}$ and $h_{\lambda\mu}$ are symmetric satisfying the equations

(3.2)a $\quad ( a_{\lambda\mu} - k\ h_{\lambda\mu} )\ \underset{i}{e}^{\lambda} = 0$

(3.2)b $\quad ( a_{\lambda\mu} - \bar{k}\ h_{\lambda\mu} )\ \underset{j}{e}^{\lambda} = 0$

then

(3.3)a $\quad h_{ij} = 0$

(3.3)b $\quad a_{ij} = 0$

(3.3)c $\quad k = a_{ii} / h_{ii}$, where $k \neq \bar{k}$.

PROOF. Multiplying both side of(3.2)a and (3.2)b by $\underset{j}{e}^{\mu}$ and $\underset{i}{e}^{\lambda}$, respectively, and subtracting the two results

(3.4) $\quad h_{\lambda\mu}\ \underset{i}{e}^{\lambda}\ \underset{j}{e}^{\mu} = 0.$

From (3.2)a and (3.4), we have(3.3)b.

Using(2.4)a and (3.2)a.

$$k = a_{\lambda\mu}\ \underset{i}{e}^{\lambda}\ \underset{j}{e}^{\mu} \ /\ h_{\lambda\mu}\ \underset{i}{e}^{\lambda}\ \underset{j}{e}^{\mu} = a_{ij}\ /\ h_{ij}.$$

# REFERENCES

1. K.T.Chung J.O.Hyun, On the nonholonomic frames of $V_n$ Yonsei Nonchong. Vol. 13, 1976.

2. J.O.hyun, On the characteristic orthogonal Nonholonomic frames, The Mathematical Education Vol. XV. No.1. 1976.

3. C.E.Weatherburn, An introduction to Riemannian Geometry and the Tensor calculus.

# ABSTRACT

On some Properties of the Nonholonomic Components in Vn

Koh,Aeja
Department of Mathematics
Graduate School of Education
Cheju National University

In our paper, we will introduce the general nonholonomic and orthogonal nonholonomic frames to an n-dimensional Riemannian space Vn, and also construct the characteristic orthogonal nonholonomic of Vn.

Finally, we will show some well-known results of Riemannian geometry in a new method using the definition and properties given.