Some of Riemannian Components on the Riemannian Manifold

이 論文을 教育學 碩士學位 論文으로 提出함.

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I. INTRODUCTION

This paper aims at calculating the components g_{ij} of the Riemannian metrics on a Riemannian manifold, the Monge patch and the upper hemisphere using the coordinate frame and imbedding, respectively as a refined way(Theorem 3.4, Corollary 3.5)

Let $\mathbf{V}$ be a vector space over $\mathbf{R}$(reals). A bilinear form

$$\Phi : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{R}$$

on $\mathbf{V}$ satisfies the conditions.

$$\Phi : (\alpha \mathbf{v}_1 + \beta \mathbf{v}_2, \ \mathbf{w}) = \alpha \Phi (\mathbf{v}_1, \ \mathbf{w}) + \beta \Phi (\mathbf{v}_2, \ \mathbf{w})$$

$$\Phi : (\mathbf{v}, \ \alpha \mathbf{w}_1 + \beta \mathbf{w}_2) = \alpha \Phi (\mathbf{v}, \ \mathbf{w}_1) + \beta \Phi (\mathbf{v}, \ \mathbf{w}_2)$$

where α, $\beta \in \mathbf{R}$ and $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{V}$, If $\dim_{\mathbf{R}}(\mathbf{V}) = n$, then a bilinear form Φ is completely determined by the n^2 values on a basis $\{\mathbf{e}_1, \ \mathbf{e}_2, \ \cdots, \ \mathbf{e}_n\}$ of $\mathbf{V}$.

We put

$$\alpha_{ij} = \Phi (\mathbf{e}_i, \ \mathbf{e}_j), \quad 1 \leqq i, \ j \leqq n$$

Then for $\mathbf{v} = \sum_{i=1}^{n} \lambda^i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^{n} \mu^j \mathbf{e}_j$, We have

$$\Phi(V,W) = \Phi\left(\sum_{i=1}^{n} \lambda^i e_i,\ \sum_{j=1}^{n} \mu^j e_j\right)$$

$$= \sum_{i=1}^{n} \lambda^i\ \Phi\left(e_i,\ \sum_{j=1}^{n} \mu^j e_j\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^i \mu^j\ \Phi(e_i, e_j)$$

$$= \sum_{i\,j=1}^{n} \alpha_{ij}\ \lambda^i \mu^i$$

If for a bilinear form on **V**, $\Phi(v,\ w) = \Phi(w,\ v)$, then Φ is said to be *symmetric*, and *skew-symmetric* if $\Phi(v,\ w) = -\Phi(w,\ v)$.

A symmetric form Φ is called a ***positive definite*** bilinear form if $\Phi(v,\ v) \geqq 0$ and $\Phi(v,\ v) = 0 \rightleftharpoons v = 0$

Throughout this paper, by a manifold we mean a differentiable C^∞-real manifold with finite dimension and U, V are open sets in $\mathbf{R}_n$

Let M be a manifold with $\dim_{\mathbf{R}}(M) = n$, and let $(U,\ \varphi)$ be a coordinate neighborhood. Then for any $p \in U$

$$\varphi : U \longrightarrow V$$

defined by $\varphi(p) = (x^1,\ x^2, \cdots,\ x^n)$ is a homeomorphism.

Ⅱ. THE TANGENT SPACE AND VECTOR FIELDS

Let M denote a C^{∞}-manifold of dimension n and let U be any open subset of M containing p, then we have defined for M the concepts of C^{∞}-function on U and C^{∞}-mapping to another manifold.

Definition 2. 1 We define *the tangent space* $T_P(M)$ to M at p to be the set of all mappings $X_P : C^{\infty}(p) \rightarrow \mathbf{R}$ satisfying for all α, $\beta \in \mathbf{R}$ and f, $g \in C^{\infty}(p)$ the two conditions

(ⅰ) $X_P(\alpha f+\beta g)=\alpha(X_P f)+\beta(X_p g)$

(ⅱ) $X_P(fg)=(X_P f)\ g(p)+f(p)(X_P g)$

with the vector space operations in $T_p(M)$ defined by

(ⅰ) $(X_p+Y_p)f=X_p f+Y_p f$

(ⅱ) $(\alpha X_p)f=\alpha(X_p f)$

A tangent vector to M at p is any $X_p \in T_p(M)$

$T(M)=\bigcup_{P\in M}T_p(M)$ is called the *tangent bundle of* M.

At each point $p \in U$, We see that if$(U,\ \varphi)$ is a coordinate neighborhood on M, then the coordinate map φ induces an isomorphism

$$
\begin{array}{cccc}
\varphi^* : & C^\infty(\varphi_{(p)}) & \longrightarrow & C^\infty(p) \\
& \Cup & & \Cup \\
& f & \longrightarrow & \varphi^*(f) = f \circ \varphi
\end{array}
$$

and an isomorphism $\varphi_* : T_p(M) \longrightarrow T_{\varphi(p)}(\mathbf{R}_n)$ of the tangent space at each point $p \in U$ onto $T_{\varphi(p)}(\mathbf{R}_n)$. On the other hand, the map φ^{-1} induces an isomorphism $\varphi_*^{-1} : T_{\varphi(p)}(\mathbf{R}_n) \longrightarrow T_p(M)$

We put

$$E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)$$

then $\{E_{1p}, E_{2p}, \cdots, E_{np}\}$ is a basis of $T_p(M)$, which is called the *coordinate frames,* where $p \in U \subset M$

Proposition 2.2 φ_* is a homomorphism.

Proof. Let X_p, $Y_p \in T_p(M)$ and f, $g \in C^\infty(p)$, Then for α, $\beta \in \mathbf{R}$

$$
\begin{aligned}
\varphi_*(\alpha X_p + \beta Y_p) f &= (\alpha X_p + \beta Y_p)(\varphi \circ f) \\
&= \alpha X_p(\varphi \circ f) + \beta Y_p(\varphi \circ f) \\
&= [\alpha \varphi_*(X_p) + \beta \varphi_*(Y_p)] f
\end{aligned}
$$

Definition 2.3 On a manifold M, *a field* Φ *of* C^∞-bilinear forms consists of a function assigning to each point $p \in M$ a bilinear form Φ_p on $T_p(M)$.

that is, a bilinear mapping

$$\Phi_p : T_p(M) \times T_p(M) \longrightarrow \mathbf{R}$$

such that for any coordinate neighborhood(U, φ), the function $\alpha_{ij} = \Phi(E_i, E_j)$ defined by Φ and coordinate frames $E_1, E_2, \cdots, E_n$ $(\dim_{\mathbf{R}}(M) = n)$ are C^r-class. The n^2 functions $\alpha_{ij} = \Phi(E_i, E_j)$ on U are called the *components of* Φ in the coordinate neighborhood (U, φ).

Definition 2.4 *A vector field X of class C^r on* M is a function assigning to each point p of M a vector $X_p \in T_p(M)$ whose components in the frames of any local coordinates(U, φ) are functions of class C^r on the domain U of the coordinates. Unless otherwise noted we will use *vector field* to mean C^∞-vector field.

Put $C^\infty(U)$ = the set of all C^∞-function on U. Let X and Y be vector fields on U. Then

(i) $\Phi(X, Y)(\forall p \in U,\ \Phi_p(X_p, Y_p))$ is of C^∞-function on U with respect to X and Y

(ii) $\forall f \in C^\infty(U),\ \Phi(fX, Y) = \Phi(X, fY) = f\Phi(X, Y)$

Ⅲ. RIEMANNIAN COMPONENTS ON THE RIEMANNIAN MANIFOLD

Suppose $F_* : \mathbf{W} \to \mathbf{V}$ is a linear map between vector spaces $\mathbf{W}$ and $\mathbf{V}$, Φ is a bilinear form on $\mathbf{V}$. Then for $\mathbf{v}, \mathbf{w} \in \mathbf{W}$, the formula

$$(F^*\Phi)(\mathbf{v}, \mathbf{w}) = \Phi(F_*\mathbf{v}, F_*\mathbf{w})$$

defines a bilinear form $F^*\Phi$ on $\mathbf{V}$.

Proposition 3.1 Under the above situation the following properties hold.

(ⅰ) If Φ is symmetric then $F^*\Phi$ is symmetric.

(ⅱ) If Φ is symmetric, positive definite and F_* is injective then $F^*\Phi$ is symmetric and positive definite.

Proof.

(ⅰ) From the above formula

$$\begin{aligned}(F^*\Phi)(\mathbf{v}, \mathbf{w}) &= \Phi(F_*\mathbf{v}, F_*\mathbf{w}) \\ &= \Phi(F_*\mathbf{w}, F_*\mathbf{v}) \\ &= F^*\Phi(\mathbf{w}, \mathbf{v})\end{aligned}$$

(ⅱ) $(F^*\Phi)(\mathbf{v}, \mathbf{w}) = \Phi(F_*\mathbf{v}, F_*\mathbf{w}) \geqq 0$

since F_* is injective

$$v=w \iff F_*v=F_*w$$

on the other hand

$$0=(F^*\Phi)(v,\ w)=\Phi(F_*v,\ F_*w)$$
$$\iff F_*v=F_*w$$
$$\iff v=w$$

Thus

$$(F^*\Phi)(v,\ w)=0 \iff v=w$$

Definition 3.2 A manifold M on which there is defined a field of symmetric, positive definite, bilinear forms Φ is called a *Riemannian manifold* and Φ is called *the Riemannian metric of* M.

Let M be a Riemannian manifold with dimension n, and let Φ be the Riemannian metric on M. For a coordinate neighborhood (U, φ) of M, We have the following definition.

Definition 3.3 We put $E_{ip}=\varphi_*^{-1}(\frac{\partial}{\partial x^i})$ $(i=1, 2, \cdots, n)$ then the n^2 functions $g_{ij}(x)=\Phi(E_{ip}, E_{jp})$ $(\varphi(p)=x \in \mathbf{R}_n)$ are called the *components of* the Rimannian metric Φ.

Let $t \longrightarrow p(t)$ $(a \leqq t \leqq b)$ be a curve of class C^1 on a Riemannian manifold M. Then the length of this curve from $\varphi(p(a))=p$ to $\varphi(p(b))=q$ is given by

$$S=L(t)=\int_a^t \Big(\sum_{i,j=1}^{n} g_{ij}(x(t)) \frac{dx^i}{dt} \frac{dx^j}{dx} \Big)^{1/2} dt$$

where $x(t)=\varphi(p(t))$. This leads to the frequently used abbreviation

$$ds^2=\sum_{i,j=1}^{n} g_{ij}(x)\, dx^i\, dx^j$$

for the Riemmanian metric Φ in local coordinates.

Thus to calculate $g_{ij}(x)$ is very important in the theory of Riemannian Geometry.

A one to one regular mapping of open set U of $\mathbf{R}^2$ into $\mathbf{R}^3$ is called a *coordinate patch* and if Ψ^{-1} is continuous then Ψ is said to be *proper patch*.

We see that if f is any differentiable real valued function on open set U in $\mathbf{R}^2$ the function $\Psi : U \longrightarrow \mathbf{R}^3$ such that $\Psi(x^1, x^2)=(x^1, x^2, f(x^1, x^2))$ is a proper patch, the patch of this type is called the *Monge patch*.

We shall calculate $g_{ij}(x)$ of the Monge patch.

Theorem 3. 4 The components of the Riemannian metric on the Monge patch is given by

$$(g_{ij})=\begin{pmatrix} 1+f_1^2, & f_1 f_2 \\ f_1 f_2, & 1+f_2^2 \end{pmatrix}$$

where $\quad f_i = \frac{\partial f}{\partial x^i}$

Proof. Suppose the formula of the Monge patch

$$\varphi : U \longrightarrow \mathbf{R}^3 \ (U : \text{open in } \mathbf{R}^2)$$

is defined by $(x^1,\ x^2) \longrightarrow (X,\ Y,\ Z) = (x^1,\ x^2,\ f(x^1,\ x^2))$

If $g_{ij} = (E_i, E_j)$ $(i,\ j = 1,\ 2)$, then

$$E_1 = \varphi_*^{-1}\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial X}\cdot 1 + \frac{\partial}{\partial Y}\cdot 0 + \frac{\partial}{\partial Z}\cdot f_1$$

$$E_2 = \varphi_*^{-1 \partial Y}\left(\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial X}\cdot 0 + \frac{\partial}{\partial X}\cdot 1 + \frac{\partial}{\partial Z}\cdot f_2$$

Thus

$$g_{11} = (E_1,\ E_1) = 1 + f^2_1$$

$$g_{12} = (E_1,\ E_2) = f_1 f_2 = g_{21}$$

$$g_{22} = (E_2,\ E_2) = 1 + f^2_2$$

Note that

(i) $(x,\ y) \neq 0$

(ii) $\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial X}\right) = \left(\frac{\partial}{\partial Y}, \frac{\partial}{\partial Y}\right) = \left(\frac{\partial}{\partial Z}, \frac{\partial}{\partial Z}\right) = 1$

$\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right) = \left(\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right) = \left(\frac{\partial}{\partial Z}, \frac{\partial}{\partial X}\right) = 0$

Consider the upper hemisphere which is one of the Monge patch, then we have the followings.

Corollary 3.5 The components of the Riemannian metric on the upper hemisphere is given by

$$(g_{ij}) = \frac{1}{1-(x^1)^2-(x^2)^2}\begin{pmatrix} 1-(x^2)^2, & x^1x^2 \\ x^1x^2, & 1-(x^1)^2 \end{pmatrix}$$

Proof. Since the formula of the upper hemisphere is given by

$$(x^1, x^2) \rightarrow (X, Y, Z) = (x^1, x^2 \sqrt{1-(x^1)^2-(x^2)^2})$$

$$E_1 = \varphi_*^{-1}\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial X}\cdot 1 + \frac{\partial}{\partial Y}\cdot 0 + \frac{\partial}{\partial Z}\cdot\frac{-x^1}{\sqrt{1-(x^1)^2-(x^2)^2}}$$

$$E_2 = \varphi_*^{-1}\left(\frac{\partial}{\partial x^2}\right) = \frac{\partial}{\partial X}\cdot 0 + \frac{\partial}{\partial Y}\cdot 1 + \frac{\partial}{\partial Z}\cdot\frac{-x^2}{\sqrt{1-(x^1)^2-(x^2)^2}}$$

Hence

$$g_{11} = (E_1, E_1) = 1 + \frac{(x^1)^2}{1-(x^1)^2-(x^2)^2}$$

$$= \frac{1-(x^2)^2}{1-(x^1)^2-(x^2)^2}$$

$$g_{12} = (E_1, E_2) = \frac{x^1x^2}{1-(x^1)^2-(x^2)^2} = g_{21}$$

$$g_{22} = (E_2, E_2) = 1 + \frac{(x^2)^2}{1-(x^1)^2-(x^2)^2}$$

$$= \frac{1-(x^1)^2}{1-(x^1)^2-(x^2)^2}$$

Thus

$$(g_{ij}) = \frac{1}{1-(x^1)^2-(x^2)^2}\begin{pmatrix} 1-(x^2)^2, & x^1x^2 \\ x^1x^2, & 1-(x^1)^2 \end{pmatrix}$$

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〈國文抄錄〉

Riemann 多樣體上에서의 Riemann 計量의 成分에 關한 小考

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본 論文에서는 첫째로 C^∞-多樣體(C^∞-Manifold)의 接空間(Tangent space)과 Vector 場(Vector field)에 대한 몇가지 性質들을 조사한다.

둘째로 Coordinate frames 을 利用하여 Riemann 多樣體인 Monge patch 와 上半球(Upper hemisphere)에서 Riemann 計量(Riemannian Metric)의 成分(Components)을 구한다.