On Constrained Minimization Problem in Hilbert space

이를 教育學碩士學位 論文으로 提出함

濟州大學校教育大學院數學教育專攻

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1986 年 6 月 日

金忠河의 碩士學位 論文을 認准함

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1986 年 6 月 日

감 사 의 글

이 논문이 완성되기 까지 바쁘신 가운데도 지도를 하여주신 현진오 교수님께 감사를 드리며, 아울러 그동안 많은 도움을 주신 김 도현 교수님과 여러 교수님께 감사를 드립니다.

그리고 그동안 저에게 격려를 하여주신 가족 및 주위의 모든 분들께 감사 드립니다.

1986년 6월 일

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1. INTRODUCTION

The operator equation $Ax=y$ where A is a mapping on some space into another has a solution if and only if y is in the range of A. This embodies the notion of a solution in the traditional sense. On the other hand, one may broaden the notion of a solution. One way to do this is to seek a solution in the least squares sense.

In 1955, Penrose[5] introduced an analytic definition of generalized inverse. In 1970, Minamide and Nakamula[3] introduced the concept of the restricted generalized inverse which possesses a "Constrained best approximation property" and which has applications to certain constrained minimization problems. Our result here is motivated by the work of Nashed[4].

The purpose of the present paper introduce the generalized inverse of linear operator and investigate the conditions under which the solution of the constrained minimization problem exists and is unique.

2. PRELIMINARIES

Let X and Y be (real or complex) Hilbert spaces and let $A: X \to Y$ be a bounded linear operator. We denote the range of A by R(A), the null space of A by N(A) and the adjoint of A by A^*. For any subspace S of a Hilbert space H, we denote by $S^\perp$ the orthogonal complement of S and the clousure of S by $\overline{S}$. Then we have the following orthogonal decomposition of X and Y (Groetsch[1])

$$X = N(A) \oplus N(A)^\perp = N(A) \oplus \overline{R(A^*)}$$

$$Y = N(A^*) \oplus N(A^*)^\perp = N(A^*) \oplus \overline{R(A)}$$

The closed range theorem is (Groetsch [1]), $R(A)$ is closed in Y if and only if $R(A^*)$ is closed in X.

Consider an operator equation of the first kind:

(2.1) $Ax = y,\ x \in X,\ y \in Y$

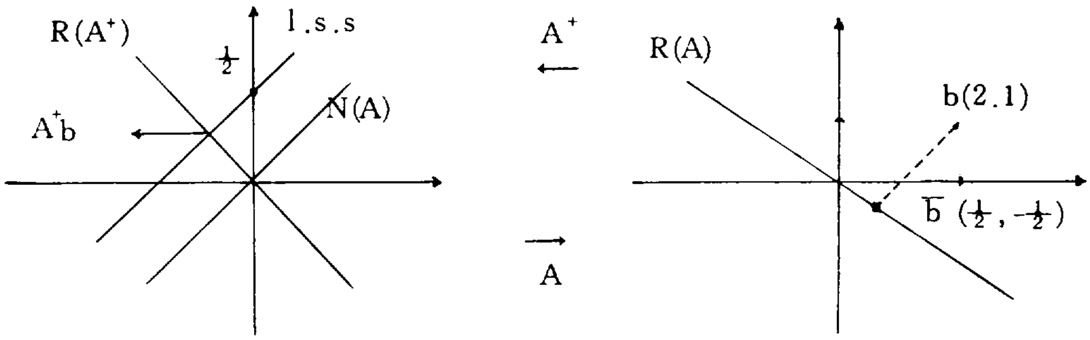
DEFINITION 2.1 For a given $y \in Y$, an element $u \in X$ is called a least squares solution of (2.1) if and only if $\|Au - y\| \leqq \|Ax - y\|$ for all $x \in X$.

DEFINITION 2.2 An element $\bar{u}$ is called a least squares solution of minimal norm of (2.1) if and only if $\bar{u}$ is a least squares solution of (2.1) and $\|\bar{u}\| \leqq \|u\|$ for all least squares solutions u of (2.1).

For each $y \in R(A) \oplus R(A)^{\perp}$ the set of all least squares solutions is nonempty closed, and convex. Hence there is a unique minimum norm solution (See Groetsch[1])

DEFINITION 2.3 Let A be a bounded linear operator form X into Y. The generalized inverse denoted by A^+ is a linear operator from the subspace $R(A) \oplus R(A)^{\perp}$ into X, defined by $A^+y = \bar{u}$, where $\bar{u}$ the least squares solution minimal norm of the equation $Ax = y$.

EXAMPLE : Let us consider the system of linear equation $A: R^2 \rightarrow R^2$ $Ax = b$ where $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ Since b is not in the range of A, this equation obviously has not solution in the traditional sense. However, by the euclidean norm let $\bar{b}$ be the point in $R(A)$ which is closet to b, then $\bar{b}$ is obviously the orthogonal projection of b on $R(A)$. Moreover, we can easily note that $R(A) = \text{span}\{(1, -1)\}$, and $\overline{b} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$. Thus, the set of all least squares solutions is given by $\{(x_1, x_2) : x_2 = \frac{1}{2} + x_1\}$ and the least squares solution of minimal norm is $(-\frac{1}{4}, \frac{1}{4})$. Hence $A^{+} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$



PROPOSITION 2.4 Let $A: X \to Y$ be a bounded linear operator and let P be the projection of Y onto $\overline{R(A)}$ Then the following conditions on $u \in X$ are equivalent

(a) $Au = Pb$

(b) $\|Au - b\| \leqq \|Ax - b\|$ for any $x \in X$

(c) $A^{*} Au = A^{*} b$

Proof. (a) $\Rightarrow$ (b) :

Suppose $Au = Pb$. Then for any $x \in X$, we have by use of the Pythagorean property and the fact that $Pb - b \in \overline{R(A)}^{\perp}$

$$\|Ax - b\|^2 = \|Ax - Pb\|^2 + \|Pb - b\|^2$$

$$= \|Ax - Pb\|^2 + \|Au - b\|^2 \geqq \|Au - b\|^2$$

(C) $\Rightarrow$ (a) :

If $A^{*} Au = A^{*} b$ then $Au - b \in \overline{R(A)}^{\perp}$ and so $0 = P(Au - b)$
$= Au - Pb$

(b) =>(C) :

Since $P_b \in \overline{R(A)}$, there is a sequence $\{X_n\}$ in X such that $P_b = \lim_n Ax_n$ and $\|b-P_b\|^2 = \lim_n \|b-Ax_n\|^2 \geqq \|b-Au\|^2$. Hence $\|Au-b\|^2 \geqq \|Au-P_b\|^2 + \|b-Au\|^2$ which gives $Au - b = P_b - b \in \overline{R(A)}^{\perp} = N(A^*)$

i.e $A^* Au = A^* b$.

PROPOSITION 2.5 A vector u is a least squares solution of(2.1) if and only if u is a solution of $A^* Ax = A^* y$

Proof. The problem of finding the least squars solntion of (2.1) is equivalent to minimizing $\|w-y\|$ over R(A).

A minimizing element $\hat{w} \in R(A)$ is obviously characterized by the condition $y - \hat{w} \in R(A)^{\perp}$ or equivalently $y - \hat{w} \in N(A^*)$, i.e. $A^* y = A^* \hat{w}$. But $\hat{w} = Au$ for some $u \in X$ and u is a least squares solution of (2.1). Thus $A^* y = A^* Au$.

DEFINITION 2.6 The operator equation (2.1) is said to be well-posed (relative to the spaces X and Y) if for each $y \in Y$, (2.1) has a unique least squares solution of minimal norm which depends continuously on y. Otherwise the equation is said to be ill-posed.

Note: When A is a linear operator with inverse. Then $A^+ = A^{-1}$ and the least squares solution of minimal norm coincides with the exact solution.

PROPOSITION 2.7 Let $A: X \to Y$ be a bounded linear operator. Then the following statements are equivalent:

(a) The operator equation (2.1) is well - posed,

(b) A has a closed range in Y,

(c) A^+ is a bounded linear operator on Y into X.

Proof. A has a closed range in Y if and only if A^+ is bounded, and if A has a closed range then $Y = R(A) \oplus R(A)^\perp = D(A^+)$ where $D(A^+)$ is the domain of A^+. Thus, we can easily check that (a), (b), (c). are equivalent.

3. CONSTRAINED MINIMIZATION PROBLEM

Let X, Y and Z be three (real or complex) Hilbert Spaces. Let $A : X \to Y$ and $L : X \to Z$ be bounded linear operators. We assume that the range R(L) of L is closed in Z, but the range R(A) of A is not necessarily closed in Y. we cosider the following minimization problem;

(3.1) For y in the domain $D(A^+)$ of A^+, let $S_y = \{u \in X : \|Au-y\|_Y = \inf \|Ax - y\|_X,\ x \in X\}$

Then the problem is to find $w \in S_y$ such that

$\|Lw\|_Z = \inf \{ \|Lu\|_Z : u \in S \}$.

In this section we state the conditions under which the solution of the problem exists and is unique.

PROPOSITION 3.1 The constrained minimization problem has a solution for every $y \in D(A^+)$ if and only if $LN(A)$ is closed.

Proof. Since for any $u \in S_Y$. $u = A_Y^+ + v$ for some $v \in N(A)$, the problem (3.1) is equivalent to $\inf \{\|L\ \| : u \in S_Y\} = \inf \{\|L(A^+_Y + v)\| : v \in N(A)\} =$

$\inf\{\|u\| : u \in LS_y\}$,

Note that LS_y is a translation of the subspace $L\,N(A)$.

Thus, we can easily check that the proposition holds.

PROPOSITION 3.2 In proposition 2.1, There exists a unique solution if and only if $N(A) \cap N(L) = \{0\}$.

Proof. ($\Leftarrow$). Suppose that $N(A) \cap N(L) = \{0\}$. Then since $N(L_A) = \{0\}$, where L_A is the restriction of L onto $N(A)$, there exists a unique $w_1 \in N(A)$ such that $\|Lw_1 + L(A_Z^+)\| \leqq \|Lx_1 + L(A_Z^+)\|$ for all $x_1 \in N(A)$.

It shows that there exists a unique $v = A_Z^+ + w_1$ such that $\|Lw\| \leqq \|Lx\|$ for all $x \in S_y$.

($\Rightarrow$) Suppose that $N(A) \cap N(L) \neq \{0\}$. Then there exists at least one $w_2 \in N(A) \cap N(L)$ which is not zero.

Thus, $\|Lw\| = \|L(w + w_2)\| \leqq \|Lx\|$ for all $x \in S_y$. Hence w is not unique.

We define a new inner product in X;

(3.2) $[u,v] = \langle Au, Av\rangle_Y + \langle Lu, Lv\rangle_Z$ for $u, v \in X$. Let $M = \{x \in X : L^* Lx \in N(A)^\perp\}$.

PROPOSITION 3.3 (a) The equation (3.1) defined an inner product in X.

(b) M is a closed subspace of X and is the orthogonal complement of N(A) with respect to the new inner product, i.e $X = N(A) \oplus_L M$ where $\oplus_L$ denotes the orthogonal decomposition with respect to $[\cdot, \cdot]$

Proof. (a) for u, v, $w \in X$ and scalar α,

$$[u+v, w] = \langle A(u+v), Aw \rangle + \langle L(u+v), Lw \rangle$$

$$= \langle Au, Aw \rangle + \langle Av, Aw \rangle = \langle Lu, Lw \rangle + \langle Lv, Lw \rangle$$

$$= [u, w] + [v, w]$$

$$[\alpha u, v] = \langle A\alpha u, Av \rangle + \langle L\alpha u, Lv \rangle$$

$$= \alpha(\langle Au, Av \rangle + \langle Lu, Lv \rangle) = \alpha[u, v]$$

$$[u, v] = \overline{[v, u]}, \qquad [u, v] = 0, \text{ iff } u = 0$$

(b) Since X is complete and M is closed, M is complete. Since M is convex, for every $x \in X$, there is a $m \in M$ such that $x = m + n$ where $n \in M^{\perp}$ (See krezig [5. p.146])

$$[n, m] = \langle An, Am \rangle_y + \langle Ln, Lm \rangle_z = \langle n, A^* Am \rangle_x + \langle n, L^* Lm \rangle_x = 0$$

Thus $\langle n, L^* Lm \rangle_x = 0$. Since $L^* Lm \in N(A)^{\perp}$,

$n \in N(A)$. Hence $X = N(A) \oplus_L M$.

We denote the space X with the inner product $[\cdot, \cdot]$ by X_L

THEOREM 3.4 An element $w \in X$ is a solution to the problem (3.1) if and only if $A^* Aw = A^* y$ and $L^* Lw \in N(A)^{\perp}$

Proof. For any $u \in S_y = \{u \in X : \|Au - y\|_Y = \inf \|Ax - y\|_Y \ x \in X\}$, $u = A^{\dagger}y + v$ for some $v \in N(A)$

Let $w \in S_y$ such that $\|Lw\|_Z \leqq \|Lu\|_Z$ for all $u \in S_y$.

Then $A^* Aw = A^* y$ by proposition 2.5 and $\|L(A^{\dagger}y + v)\| \leqq \|L(A^{\dagger}y + x_1)\|$ for all $x_1 \in N(A)$ …… (*) where $w = A^{\dagger}y + v$.

Note that N(A) is a closed subspace of X. Now, consider the restriction of

L onto N(A), denoted by LA. Since $Z = R(\overline{LA}) \oplus R(LA)^{\perp}$ the above condition(*) is equivalent to $L(A^{+}y + v) \in R(LA)^{\perp}$.

Thus for all $x \in N(A)$, $(Lx, L(A^{+}y + v)) = 0$ if and only if $(x, L^{*}L(A^{+}y + v)) = 0$ for all $x \in N(A)$. Namely $L^{*}Lw \in N(A)^{\perp}$.

By this theorem, the problem of constrained minimization (3.1) is equivalent to finding an element $w \in M$ such that $A^{*}Aw = A^{*}y$.

Thus the solution w is the least squares solution of X_L-minimal norm of the equation(2.1).

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(國文抄錄)

Hilbert 空間上에서 制限된 Minimization 問題에 대한 硏究

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(指導敎授 玄 進 五)

본 論文에서는 線型作用小의 一般逆을 紹分하고, Hilbert 空間上에서 有限 線型作用小가 閉領域을 가지면서 그 作用小를 制限하는 또 다른 有限作用小가 주어졌을때 그 線型作用小의 最小化 問題에 대한 解의 樣相을 알아 보았다.