

## AN ASYMPTOTIC STABILITY AND A UNIFORM ASYMPTOTIC STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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**ABSTRACT.** We consider a system of functional differential equation  $x'(t) = F(t, x_t)$  and obtain conditions on a Liapunov functional to ensure the asymptotic stability and the uniform asymptotic stability of the zero solution.

### 1. INTRODUCTION

The purpose of this paper is to present sufficient conditions, using Liapunov's direct method, to ensure that the zero solution of a system of functional differential equations with infinite delay (including finite delay) is asymptotically stable and that the zero solution of a system of functional differential equations with finite delay is uniformly asymptotically stable. This is, of course, an old problem, and there are many well-known results and applications.

We consider the system

$$(1) \quad x'(t) = F(t, x_t),$$

where  $x_t$  is the translation of  $x$  on  $[t-h, t]$  back to  $[-h, 0]$ , where  $h > 0$  is a fixed constant, and  $x'$  denotes the right-hand derivative. The following notation will be used.

For  $x \in R^n$ ,  $|x|$  denotes a usual norm in  $R^n$ . For  $h > 0$ ,  $C$  denotes the space of continuous functions mapping  $[-h, 0]$  into  $R^n$ , and, for  $\phi \in C$ ,  $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)|$ . Also,  $C_H$  denotes the set of  $\phi \in C$  with  $\|\phi\| < H$ . If  $x$  is a continuous function of  $u$  defined for  $-h \leq u < A$ , with  $A > 0$ , and if  $t$  is a fixed number satisfying  $0 \leq t < A$ , then  $x_t$  denotes the restriction of  $x$  to  $[t-h, t]$  so that  $x_t$  is an element of  $C$  defined by  $x_t(\theta) = x(t-\theta)$  for  $-h \leq \theta \leq 0$ . We denote by  $x(t_0, \phi)$  a solution of (1) with initial condition  $\phi \in C$  where  $x_{t_0}(t_0, \phi) = \phi$ , and we denote by  $x(t, t_0, \phi)$  the value of  $x(t_0, \phi)$  at  $t$ .

It is supposed that  $F: R_+ \times C_H \rightarrow R^n$  is continuous and takes bounded sets into bounded sets; where  $0 < H \leq \infty$ . It is well known [6, 10] that for each  $t_0 \in R_+ = [0, \infty)$  and each  $\phi \in C_H$  there is at least one solution

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$x(t_0, \phi)$  defined on an interval  $[t_0, t_0 + \alpha)$  and, if there is an  $H_1 < H$  with  $|x(t, t_0, \phi)| \leq H_1$ , then  $\alpha = \infty$ .

A Liapunov functional is a continuous  $V(t, \phi): R_+ \times C_H \rightarrow R_+$  whose derivative along a solution of (1) satisfies some specific relation. The derivative of a Liapunov functional  $V(t, \phi)$  along a solution  $x(t)$  of (1) may be defined in several equivalent ways. If  $V$  is differentiable, the natural derivative is obtained using the chain rule. But, in general,  $V'_{(1)}(t, \phi)$  denotes the derivative of functional  $V$  with respect to (1) defined by

$$V'_{(1)}(t, \phi) = \limsup_{\delta \rightarrow 0^+} \{V(t + \delta, x_{t+\delta}(t, \phi)) - V(t, \phi)\} / \delta.$$

**Definition 1.** Let  $H > 0$ ,  $S_H = \{x \in R^n \mid |x| < H\}$ , and let  $U: R_+ \times S_H \rightarrow R$  be continuous and locally Lipschitz in  $x$ . Then the derivative of  $U(t, x)$  along a solution  $x$  of (1) is defined as

$$U'_{(1)}(t, x) = \limsup_{\delta \rightarrow 0^+} \{U(t + \delta, x + \delta F(t, x_t)) - U(t, x)\} / \delta.$$

*Remark 1.* (i) It is easy to check that

$$\limsup_{\delta \rightarrow 0^+} \frac{1}{\delta} \{U(t + \delta, x(t + \delta)) - U(t, x(t))\} = U'_{(1)}(t, x(t))$$

for any solution  $x(t)$  of (1).

(ii) If  $U(t, x(t))$  has continuous partial derivatives of the first order,

$$U'_{(1)}(t, x(t)) = \text{grad } U \cdot F + \partial U / \partial t.$$

**Definition 2.** Let  $F(t, 0) = 0$ , for all  $t \geq 0$ .

(a) The zero solution of (1) is said to be stable if for each  $\varepsilon > 0$  and  $t_0 \geq 0$  there is a  $\delta > 0$  such that  $[\phi \in C_\delta, t \geq t_0]$  imply  $|x(t, t_0, \phi)| < \varepsilon$ .

(b) The zero solution is uniformly stable (U.S.) if it is stable and if  $\delta$  is independent of  $t_0$ .

(c) The zero solution is asymptotically stable (A.S.) if it is stable and if for each  $t_0 \geq 0$  there is a  $\delta > 0$  such that  $\phi \in C_\delta$  implies that  $x(t, t_0, \phi) \rightarrow 0$  as  $t \rightarrow \infty$ .

(d) The zero solution is uniformly asymptotically stable (U.A.S.) if it is U.S. and if there is an  $\eta > 0$  and for each  $\gamma > 0$  there exists  $T > 0$  such that  $[t_0 \in R_+, \phi \in C_\eta, t \geq t_0 + T]$  imply that  $|x(t, t_0, \phi)| < \gamma$ .

**Definition 3.** A measurable function  $\eta: R_+ \rightarrow R_+$  is said to be integrally positive with parameter  $\delta > 0$  (IP( $\delta$ )) if whenever  $I = \bigcup_{m=1}^{\infty} [\alpha_m, \beta_m]$  with  $\alpha_m < \beta_m < \alpha_{m+1}$  and  $\beta_m - \alpha_m \geq \delta$  ( $m = 1, 2, 3, \dots$ ), then  $\int_I \eta(t) dt = \infty$ . If a function  $\eta$  is integrally positive for every  $\delta > 0$ , then it is called integrally positive (IP).

**Definition 4.** Let  $\eta: R_+ \rightarrow R_+$  be measurable.

(a) The function  $\eta$  is said to be weakly integrally positive with parameters  $\delta > 0$  and  $\Delta > 0$  (WIP( $\delta, \Delta$ )) if whenever  $\{t_i\}$  and  $\{\delta_i\}$  satisfy  $t_i + \delta_i < t_{i+1} \leq t_i + \delta_i + \Delta$  with  $\delta_i \geq \delta$ , then

$$\sum_{i=1}^{\infty} \int_{t_i}^{t_i + \delta_i} \eta(t) dt = \infty.$$

(b) The function  $\eta$  is said to be uniformly weakly integrally positive with parameters  $\delta > 0$  and  $\Delta > 0$  (UWIP( $\delta, \Delta$ )) if (a) holds and for every  $M > 0$  there exists  $Q > 0$  such that for all  $S > Q$  and for all  $\{t_i\}$  and  $\{\delta_i\}$  satisfying (a), then

$$\int_{[t_1, t_1+S] \cap I} \eta(t) dt > M \quad \text{where } I = \bigcup_{i=1}^{\infty} [t_i, t_i + \delta_i].$$

*Remark 2.* If  $\eta$  is IP( $\delta$ ), then it is UWIP( $\delta, \Delta$ ) for all  $\Delta > 0$ . The converse is false. See [4, Remark 4].

In presenting sufficient conditions, the following theorem is basic. Denote by  $W_i$  the continuous functions from  $R_+ \rightarrow R_+$ ,  $W_i(0) = 0$ , and  $W_i(\gamma)$  strictly increasing (called wedges).

**Theorem A** (see [6, p. 105]). Let  $H > 0$  and  $V: R_+ \times C_H \rightarrow R_+$  be continuous. If  $\exists$  wedges  $W_1, W_2, W_3$  such that, for all  $\phi \in C_H$ ,

- (i)  $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ ,
- (ii)  $V'_{(1)}(t, x_t) \leq -W_3(|x(t)|)$ , and
- (iii)  $F(t, \phi)$  is bounded for  $\phi$  bounded.

then  $x = 0$  of (1) is uniformly asymptotically stable.

One of our main goals is to eliminate condition (iii) in the above theorem.

## 2. MAIN RESULTS AND SOME REMARKS

Let  $|x(t)|'$  be the right-hand derivative of  $|x(t)|$ , let  $\{a(t)\}_+ = \max\{a(t), 0\}$ , and let  $\{a(t)\}_- = \max\{-a(t), 0\}$ .

**Theorem 1.** Let  $H > 0$  and  $V: R_+ \times C_H \rightarrow R_+$  be continuous and locally Lipschitzian in  $\phi$  and  $\eta$  be WIP( $\beta, \Delta$ ) for any  $\beta > 0$  and  $\Delta > 0$ . Suppose  $U: R_+ \times R^n \rightarrow R_+$  is continuous and locally Lipschitz in  $x$  such that either  $\int_0^t \{U'(s, x)\}_+ ds$  or  $\int_0^t \{U'(s, x)\}_- ds$  is uniformly continuous for any bounded solution  $x(t)$  of (1) on  $R_+$ . Further, suppose  $\exists$  wedges  $W_1, W_2, W_3$ , and  $W_4$  such that, for all  $t \geq 0$  and  $\phi$  in  $C_H$ ,

- (i)  $W_1(|\phi(0)|) \leq V(t, \phi)$  and  $V(t, 0) = 0$ ,
- (ii)  $V'_{(1)}(t, x_t) \leq -\eta(t)W_2(|x(t)|)$ , and
- (iii)  $W_3(|x(t)|) \leq U(t, x(t)) \leq W_4(|x(t)|)$ .

Then the zero solution of (1) is asymptotically stable.

*Proof.* It is evident that the zero solution is stable. By stability, there is a  $\delta = \delta(t_0, H)$  such that  $[t_0 \geq 0, \phi \in C_\delta, t \geq t_0]$  imply that  $|x(t, t_0, \phi)| < H$ . Suppose that for some such  $(t_0, \phi)$  the solution  $x(t) = x(t, t_0, \phi) \rightarrow 0$  as  $t \rightarrow \infty$ . First we claim that  $\liminf_{t \rightarrow \infty} |x(t)| = 0$ . If this is false, then there exist constants  $\theta, T > 0$  such that  $|x(t)| \geq \theta$ , for  $t \geq t_0 + T$ . Thus

$$\lim_{t \rightarrow \infty} V(t, x_t) \leq V(t_0, \phi) - W_2(\theta) \int_{t_0+T}^{\infty} \eta(s) ds = -\infty,$$

a contradiction. Suppose that  $\int_0^t \{U'(s, x(s))\}_+ ds$  is uniformly continuous on  $R_+$ . Then for some  $\gamma > 0$ , we can choose a constant  $\theta > 0$  and a sequence  $t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n < \dots$  such that  $W_4(\theta) < W_3(\gamma)$

and, for  $i = 1, 2, 3, \dots$ ,  $|x(\alpha_i)| = \theta$ ,  $|x(\beta_i)| = \gamma$ , and  $\theta \leq |x(t)|$ , for any  $t \in [\alpha_i, \beta_i]$ . Thus we have

$$\begin{aligned} W_3(|x(\beta_i)|) &\leq U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'(s, x(s)) ds + U(\alpha_i, x(\alpha_i)) \\ &\leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+ ds + W_4(|x(\alpha_i)|) \end{aligned}$$

and

$$0 < W_3(\gamma) - W_4(\theta) \leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+ ds.$$

By assumption there exists  $\rho > 0$  such that  $\beta_i - \alpha_i \geq \rho$  for  $i = 1, 2, 3, \dots$ . Let  $I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$ . Then we have

$$\lim_{t \rightarrow \infty} V(t, x_t) \leq V(t_0, \phi) - W_2(\theta) \int_I \eta(s) ds = -\infty,$$

a contradiction. Suppose that  $\int_0^t \{U'(s, x(s))\}_- ds$  is uniformly continuous on  $R_+$ . Then for some  $\gamma > 0$ , we can choose a constant  $\theta > 0$  and a sequence  $t_i < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_i < \beta_i < \dots$  such that  $W_4(\theta) < W_3(\gamma)$  and, for  $i = 1, 2, 3, \dots$ ,  $|x(\alpha_i)| \geq \gamma$ ,  $|x(\beta_i)| = \theta$ , and  $\theta \leq |x(t)|$ , for any  $t \in [\alpha_i, \beta_i]$ . Thus we have

$$\begin{aligned} W_4(|x(\beta_i)|) &\geq U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'(s, x(s)) ds + U(\alpha_i, x(\alpha_i)) \\ &\geq - \int_{\alpha_i}^{\beta_i} U'(s, x(s))_- ds + W_3(|x(\alpha_i)|) \end{aligned}$$

and

$$0 < W_3(\gamma) - W_4(\theta) \leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_- ds.$$

By assumption there exists  $\rho > 0$  such that  $\beta_i - \alpha_i \geq \rho$  for  $i = 1, 2, 3, \dots$ . Let  $I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$ . Then we have

$$\lim_{t \rightarrow \infty} V(t, x_t) \leq V(t_0, \phi) - W_2(\theta) \int_I \eta(s) ds = -\infty,$$

a contradiction. Thus the proof is complete.

*Remark 3.* The condition that either  $\int_0^t \{U(s, x(s))\}_+ ds$  or  $\int_0^t \{U(s, x(s))\}_- ds$  is uniformly continuous for any bounded solution  $x(t)$  of (1) on  $R_+$  is satisfied if

$$-p(t) \leq U'(t, x(t)) \quad \text{or} \quad U'(t, x(t)) \leq q(t),$$

where  $p, q: R_+ \rightarrow R_+$  are measurable functions such that  $\int_0^t p(s) ds$  and  $\int_0^t q(s) ds$  are uniformly continuous on  $R_+$ .

**Corollary 1.** Let  $V: R_+ \times C_H \rightarrow R_+$  be continuous and let  $\eta$  be WIP( $\delta, \Delta$ ) for any  $\delta > 0$  and  $\Delta > 0$ . Suppose that

- (i)  $W_1(|x(t)|) \leq V(t, x_t)$  and  $V(t, 0) = 0$ ,
- (ii)  $V'_{(1)}(t, x_t) \leq -\eta(t)W_2(|x(t)|)$ , and
- (iii)  $F(t, \phi)$  is bounded for  $\phi$  bounded.

Then the zero solution of (1) is asymptotically stable.

Now, we consider a system of functional differential equations with unbounded delay

$$(2) \quad x' = F(t, x(s); \alpha \leq s \leq t), \quad -\infty \leq \alpha.$$

To specify a solution of (2) we require a  $t_0 \geq \alpha$  and a bounded continuous function  $\phi: [\alpha, t_0] \rightarrow R^n$ ; we then obtain a solution  $x(t, t_0, \phi)$  satisfying (2) on an interval  $[t_0, t_0 + \beta]$  with  $x(t, t_0, \phi) = \phi(t)$  for  $\alpha \leq t \leq t_0$ . For details see Driver [5] or Burton [2]. To make the presentation here parallel that for finite delay equations, for each  $t > \alpha$  we consider the function space  $C(t)$  with  $\phi \in C(t)$  if  $\phi: [\alpha, t] \rightarrow R^n$  is bounded and continuous. The norm used is the supremum norm  $\|\cdot\|$ . Thus, for any  $t_0 > \alpha$ , our initial function is some  $\phi \in C(t_0)$  and our definitions of stability coincide with the one for finite delay. A Liapunov functional is denoted by  $V(t, x(\cdot))$ . For convenience we may assume that  $t_0 \geq 0$ .

**Theorem 2.** Let  $H > 0$  and for each  $t_0 > \alpha$  let  $C_H(t_0) \subset C(t_0)$  with  $\phi \in C_H(t_0)$  if  $\|\phi\| < H$ , and let  $V: [t_0, \infty) \times C_H(t_0) \rightarrow R_+$  be continuous and locally Lipschitz in  $\phi$  and  $\eta$  be WIP( $\beta, \Delta$ ) for any  $\beta > 0$  and  $\Delta > 0$ . Suppose  $U: R_+ \times R^n \rightarrow R_+$  is continuous and locally Lipschitz in  $x$  such that either  $\int_0^t \{U'(s, x)\}_+ ds$  or  $\int_0^t \{U'(s, x)\}_- ds$  is uniformly continuous for any bounded solution  $x(t)$  of (1) on  $R_+$ . Further, suppose  $\exists$  wedges  $W_1, W_2, W_3$ , and  $W_4$  such that, for all  $t \geq t_0$  and  $\phi$  in  $C_H(t_0)$ ,

- (i)  $W_1(|\phi(0)|) \leq V(t, \phi)$  and  $V(t, 0) = 0$ ,
- (ii)  $V'_{(1)}(t, x_t) \leq -\eta(t)W_2(|x(t)|)$ , and
- (iii)  $W_3(|x(t)|) \leq U(t, x(t)) \leq W_4(|x(t)|)$ .

Then the zero solution of (2) is asymptotically stable.

*Proof.* The proof requires only slight modifications of the proof of Theorem 1.

**Example 1.** Consider the scalar equation

$$(A) \quad x'(t) = -a(t)x(t) + b(t)x(t - \lambda t),$$

where  $a: R_+ \rightarrow R_+$  is continuous,  $b: R_+ \rightarrow R$  is continuous,  $|b(t - \lambda t)| \geq |b(t)|$  with  $0 < \lambda < 1$ , and  $\eta(t) = a(t) - |b(t)|/(1 - \lambda)$  is WIP( $\delta, \Delta$ ) for any  $\delta > 0$  and  $\Delta > 0$ . Then the zero solution of (A) is A.S.

*Proof.* Consider the functional

$$V(t, x_t) = |x(t)| + \frac{1}{1 - \lambda} \int_{(1 - \lambda)t}^t |b(s)| |x(s)| ds.$$

Then we have

$$\begin{aligned} V'(t, x_t) &\leq -a(t)|x(t)| + |b(t)| |x(t - \lambda t)| \\ &\quad + \frac{1}{1 - \lambda} |b(t)| |x(t)| - |b(t - \lambda t)| |x(t - \lambda t)| \\ &\leq -\left(a(t) - \frac{1}{1 - \lambda} |b(t)|\right) |x(t)| - (|b(t - \lambda t)| - |b(t)|) |x(t - \lambda t)| \\ &\leq -\left(a(t) - \frac{1}{1 - \lambda} |b(t)|\right) |x(t)|. \end{aligned}$$

Since  $|b(t - \lambda t)| \geq |b(t)|$  for any  $t \geq 0$  with  $0 < \lambda < 1$ ,  $|b(t)|$  is bounded on  $R_+$ . Also, we have

$$U'(t, x(t)) = |x(t)|' \leq -a(t)|x(t)| + |b(t)||x(t - \lambda t)|.$$

That is,  $U(t, x(t))$  is bounded above on  $R_+$ , where  $U(t, x(t)) = |x(t)|$ . Hence, it follows from Theorem 2 that the zero solution of (A) is A.S.

**Example 2.** Consider the scalar equation

$$(B) \quad x'(t) = -a(t)f(x(t)) + \int_{-\infty}^t C(t-s)g(x(s)) ds,$$

where  $a: R_+ \rightarrow R_+$  is continuous,  $C: R_+ \rightarrow R$  is continuous with  $\int_0^\infty |C(u)| du < \infty$ .  $\eta(t) = a(t) - M \int_0^\infty |C(u)| du$  is  $WIP(\delta, \Delta)$  for any  $\delta > 0$  and  $\Delta > 0$ .  $f: R \rightarrow R$  is continuous and strictly increasing with  $f(0) = 0$ , and  $g: R \rightarrow R$  is continuous with  $|g(x)| \leq M|f(x)|$  for some  $M \geq 0$ . Then  $x = 0$  of (B) is A.S.

*Proof.* Consider the functional

$$V(t, x(\cdot)) = |x(t)| + \int_{-\infty}^t \int_t^\infty |C(u-s)| du |g(x(s))| ds.$$

Then we have

$$\begin{aligned} V'(t, x(\cdot)) &\leq -a(t)|f(x(t))| + \int_{-\infty}^t |C(t-s)||g(x(s))| ds \\ &\quad + \int_t^\infty |C(u-t)| du |g(x(t))| - \int_{-\infty}^t |C(t-s)||g(x(s))| ds \\ &\leq -a(t)|f(x(t))| + M \int_0^\infty |C(u)| du |f(x(t))| \\ &= -\left\{a(t) - M \int_0^\infty |C(u)| du\right\} |f(x(t))|. \end{aligned}$$

Also, we have

$$\begin{aligned} U'(t, x(t)) &= |x(t)|' \leq -a(t)|f(x(t))| + \int_{-\infty}^t |C(t-s)||g(x(s))| ds \\ &\leq -a(t)|f(x(t))| + M \int_0^\infty |C(u)| du f(\|x_t\|), \end{aligned}$$

where  $\|x_t\| = \sup_{-\infty \leq s \leq t} |x(s)|$ . That is,  $|x(t)|'$  is bounded above. Thus, it follows from Theorem 2 that  $x = 0$  of (B) is A.S.

Now, we prove a uniform asymptotic stability theorem for a system of functional differential equations with finite delay.

**Theorem 3.** Let  $H > 0$  and  $V: R_+ \times C_H \rightarrow R_+$  be continuous and locally Lipschitzian in  $\phi$  and  $\eta$  be  $UWIP(\beta, h)$  for any  $\beta > 0$ . Suppose  $U: R_+ \times R^n \rightarrow R_+$  is continuous and locally Lipschitz in  $x$  such that either  $\int_0^t \{U'(s, x)\}_+ ds$  or  $\int_0^t \{U'(s, x)\}_- ds$  is uniformly continuous for any bounded solution  $x(t)$  of (1) on  $R_+$ . Further, suppose  $\exists$  wedges  $W_1, W_2, W_3, W_4$ , and  $W_5$  such that, for all  $t \geq 0$  and  $\phi$  in  $C_H$ ,

$$(i) \quad W_1(\|\phi(0)\|) \leq V(t, \phi) \leq W_2(\|\phi\|),$$

- (ii)  $V'_{(1)}(t, x_t) \leq -\eta(t)W_3(|x(t)|)$ , and  
 (iii)  $W_4(|x(t)|) \leq U(t, x(t)) \leq W_5(|x(t)|)$ .

Then the zero solution of (1) is uniformly asymptotically stable.

*Proof.* It is evident that the zero solution is U.S. Let  $0 < H' < H$  and take  $\delta_0 = \delta_0(H')$  of U.S. For any  $\varepsilon > 0$ , we try to show that there is a  $T = T(\varepsilon) > 0$  such that any solution  $x(t, t_0, \phi)$  of (1) with  $\|\phi\| < \delta_0$  satisfies  $|x(t, t_0, \phi)| < \varepsilon$  for any  $t \geq t_0 + T$ . Let  $\delta = \delta(\varepsilon)$  be the above constant for uniform stability. Suppose that a solution  $x = x(t, t_0, \phi)$ ,  $\|\phi\| < \delta_0$ , satisfies  $\|x_t(t_0, \phi)\| \geq \delta$ , for any  $t \geq t_0$ . Then we have  $t^* \in [t, t+h]$  for each  $t \geq t_0$  such that  $|x(t^*)| \geq \delta$ . Now we can choose a constant  $\theta > 0$  with  $W_4(\delta) > W_5(\theta)$ . By assumption on  $\eta$  there is an  $L = L(\varepsilon) > 0$  such that there is at least  $t' \in [t, t+L]$  with  $|x(t')| \leq \theta$  for any  $t \geq t_0$ .

By Definition 4 there is an  $L = L(\varepsilon) > 0$  such that

$$\int_{t_0}^{t_0+L} \eta(s) ds > W_2(\delta_0)/W_3(\theta).$$

If  $|x(t)| > \theta$  were true for all  $t \in [t_0, t_0 + L]$ , then we would have

$$\begin{aligned} 0 &\leq V(t_0 + L) \leq V(t_0, \phi) - \int_{t_0}^{t_0+L} \eta(s)W_3(|x(s)|) ds \\ &\leq W_2(\delta_0) - W_3(\theta) \int_{t_0}^{t_0+L} \eta(s) ds < 0, \end{aligned}$$

a contradiction.

Now, we shall assume that  $\int_0^t \{U'(s, x(s))\}_+ ds$  is uniformly continuous on  $R_+$  and  $L \geq h$ . Then we can choose a sequence

$$t_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_i < \beta_i < \dots$$

such that, for  $i = 1, 2, 3, \dots$ ,

$$|x(\alpha_i)| = \theta, \quad |x(\beta_i)| \geq \delta, \quad \theta \leq |x(t)| \quad \text{for any } t \in [\alpha_i, \beta_i],$$

$\alpha_i \in [t_0 + (2i-2)h, t_0 + (2i-1)h] \cup I_i$ , and  $\beta_i \in I_i$ , where  $I_i = [t_0 + (2i-1)h, t_0 + 2ih]$ . Thus we have

$$\begin{aligned} W_4(|x(\beta_i)|) &\leq U(\beta_i, x(\beta_i)) = \int_{\alpha_i}^{\beta_i} U'(s, x(s)) ds + U(\alpha_i, x(\alpha_i)) \\ &\leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+ ds + W_5(|x(\alpha_i)|) \end{aligned}$$

and

$$0 < W_4(\delta) - W_5(\theta) \leq \int_{\alpha_i}^{\beta_i} U'(s, x(s))_+ ds.$$

By assumption there exists a  $\rho > 0$  with  $\beta_i - \alpha_i \geq \rho$  for  $i = 1, 2, 3, \dots$ . Let  $\mu = \min\{h, \rho\}$  and  $I = \bigcup_{i=1}^{\infty} [\beta_i - \mu, \beta_i]$ . Then we have

$$\lim_{t \rightarrow \infty} V(t, x_t) \leq V(t_0, \phi) - W_3(\theta) \int_I \eta(s) ds = -\infty,$$

a contradiction. Let  $N$  be the smallest positive integer such that

$$W_2(\delta_0) - W_3(\theta) \sum_{i=1}^N \int_{\beta_i - \mu}^{\beta_i} \eta(s) ds < 0.$$

Then  $N$  only depends on  $\varepsilon$  and we can take  $T = 2Nh$  such that, at some  $\tau \in [t_0, t_0 + T]$   $\|x_\tau(t_0, \phi)\|$ . Thus the proof is complete.

**Example 3.** Consider the scalar equation

$$(C) \quad x'(t) = -a(t)x(t) + b(t)x(t-h),$$

where  $a: R_+ \rightarrow R_+$  is continuous,  $b: R_+ \rightarrow R$  is continuous,  $\int_0^t |b(s)| ds$  is uniformly continuous on  $R_+$ , and  $\eta(t) = a(t) - |b(t+h)|$  is UWIP( $\delta, h$ ) for  $\delta > 0$ . Then  $x = 0$  of (C) is U.A.S.

*Proof.* Consider the functional

$$V(t, x_t) = |x(t)| + \int_{t-h}^t |b(u+h)| |x(u)| du.$$

Then we have

$$\begin{aligned} |x(t)| &\leq V(t, x_t) = |x(t)| + \int_{t-h}^t |b(u+h)| |x(u)| du \\ &\leq \|x_t\| + \|x_t\| \int_{t-h}^t |b(u+h)| du \\ &\leq \|x_t\| + M \|x_t\| \quad \text{for some } M > 0 \text{ with } \int_{t-h}^t |b(u+h)| du \leq M \end{aligned}$$

on  $R_+$  (by Remark 6). Also, we have

$$\begin{aligned} V'(t, x_t) &\leq -a(t)|x(t)| + |b(t)||x(t-h)| + |b(t+h)||x(t)| - |b(t)||x(t-h)| \\ &\leq -\{a(t) - |b(t+h)|\}|x(t)|. \end{aligned}$$

Furthermore,

$$U'(t, x(t)) = |x(t)|' \leq -a(t)|x(t)| + |b(t)||x(t-h)| \leq |b(t)||x(t-h)|$$

and

$$\int_0^t |b(s)| ds \quad \text{is uniformly continuous on } R_+.$$

Therefore, it follows from Theorem 3 that  $x = 0$  of (C) is U.A.S.

**Remark 3.** The above example generalizes the results of the example in [9].

**Remark 4.** Consider the scalar equation

$$(D) \quad x'(t) = -a(t)x(t) + b(t)x(t-\gamma(t)),$$

where  $a(t): R_+ \rightarrow R_+$  is continuous,  $\gamma(t) \geq 0$  with  $\gamma'(t) \leq \alpha < 1$  and  $\gamma(t) < h$  for some  $h > 0$ ,  $b(t): R_+ \rightarrow R$  is continuous with  $|b(t-\gamma(t))| \geq |b(t)|$ ,  $a_0(1-\alpha) \geq 1$  for some  $a_0 > 0$ , and  $\eta(t) = a(t) - a_0|b(t)|$  is UWIP( $\delta, h$ ) for any  $\delta > 0$ . Then  $x = 0$  of (D) is U.A.S.



*Proof.* Consider the functional

$$V(t, x(\cdot)) = |x(t)| + a_0 \int_{t-\gamma(t)}^t |b(s)| |x(s)| ds.$$

Then we have

$$\begin{aligned} V'(t, x(\cdot)) &\leq -a(t)|x(t)| + |b(t)| |x(t-\gamma(t))| + a_0 |b(t)| |x(t)| \\ &\quad - a_0(1-\alpha) |b(t-\gamma(t))| |x(t-\gamma(t))| \\ &\leq -\{a(t) - a_0 |b(t)|\} |x(t)|. \end{aligned}$$

Since

$$U'(t, x(t)) = |x(t)|' \leq -a(t)|x(t)| + |b(t)| |x(t-\gamma(t))|$$

and  $|b(t)|$  is bounded,  $|x(t)|'$  is bounded above. Thus  $x = 0$  of (D) is U.A.S.

Now, we shall compare Theorem 2 with the following theorem proved by Burton and Hatvani (see [4]).

**Definition 5.** A measurable function  $\eta: R_+ \rightarrow R_+$  is said to be positive in measure (PIM) if for every  $\varepsilon > 0$  there are  $T \in R_+$  and  $\delta > 0$  such that  $[t \geq T, Q \subset [t-h, t]$  is open,  $\mu(Q) \geq \varepsilon$ ] imply that  $\int_Q \eta(t) dt \geq \delta$ .

**Theorem B.** Let  $\eta$  be PIM and  $V: R_+ \times C_H \rightarrow R_+$  be continuous with

- (i)  $W_1(|\phi(\cdot)|) \leq V(t, \phi) \leq W_2(|\phi(\cdot)|) + W_3(\|\phi\|)$  and
- (ii)  $V'(t, x_t) \leq -\eta(t)W_4(|x(t)|)$ ,

where  $\|\phi\| = [\int_{-h}^0 |\phi(s)|^2 ds]^{1/2}$ . Then  $x = 0$  of (1) is U.A.S.

**Remark 4.** We obtain more general results when we apply Theorem 3 to Example 5 than when we apply Theorem B to Example 5, because the PIM condition is stronger than the UWIP( $\delta, h$ ) condition for any  $\delta > 0$  (cf. Theorem 11 and Remark 4 in [4]) and  $|b(t)|$  should be bounded in order to have  $V'(t, x_t) \leq -\eta(t)W(|x(t)|)$  for some coefficient function  $\eta$  and wedge  $W$ .

**Example 5.** Consider the scalar equation

$$(E) \quad x'(t) = -a(t)x(t) + b(t) \int_{t-h}^t x(u) du,$$

where  $a(t): R_+ \rightarrow R_+$  is continuous,  $b(t): R_+ \rightarrow R$  is continuous,  $\int_t^{t+h} |b(s)| ds$  is bounded on  $R_+$ , and  $a(t) - \int_t^{t+h} |b(s)| ds$  is PIM.

By Theorem B the zero solution of (E) is U.A.S. (see [4]), but we can also show that  $x = 0$  of (E) is U.A.S. under the condition that  $a(t) - \int_t^{t+h} |b(s)| ds$  is UWIP( $\delta, h$ ) for any  $\delta > 0$  and  $\int_0^t |b(s)| ds$  is uniformly continuous on  $R_+$ .

*Proof.* We can use the functional, which was used in an example in [4] and we note that  $\int_t^{t+h} |b(s)| ds$  is bounded on  $R_+$ . For details see [4].

**Remark 5.** Theorem 2 generalizes Theorem 2.1 in [6, p. 105] and Theorem 8.7.4 in [2, p. 301].

**Remark 6.** If  $b(t): R_+ \rightarrow R_+$  is measurable on  $R_+$  and  $\alpha(t) = \int_0^t b(s) ds$  is uniformly continuous on  $R_+$ , then  $f(t) = \int_t^{t+b} b(s) ds$  is bounded on  $R_+$  for some  $\delta > 0$ .

*Proof.* Since  $\alpha(t) = \int_0^t b(s) ds$  is uniformly continuous on  $[0, \infty)$ , there exists  $\delta^* = \delta^*(\delta) > 0$  such that  $\int_t^{t+\delta^*} b(s) ds \leq \delta$  for any  $t \geq 0$ . Now we can choose the smallest positive integer  $N$  such that  $N\delta^* \geq \delta$ . Then we have

$$f(t) = \int_t^{t+\delta} b(s) ds \leq \int_t^{t+N\delta^*} b(s) ds \leq N\delta \quad \text{for any } t \geq 0.$$

**Remark 7.** If  $b(t): R_+ \rightarrow R_+$  is bounded and continuous on  $R_+$ , then, obviously,  $\alpha(t) = \int_0^t b(s) ds$  is uniformly continuous on  $R_+$ . If  $b(t): R_+ \rightarrow R_+$  is unbounded and continuous on  $R_+$  and  $\int_0^\infty b(s) ds < \infty$ , then  $\alpha(t) = \int_0^t b(s) ds$  is uniformly continuous on  $R_+$  (by [8, Proposition 14, p. 88]). But the next example shows that  $\alpha(t) = \int_0^t b(s) ds$  is uniformly continuous on  $R_+$  even though  $b(t): R_+ \rightarrow R_+$  is unbounded, continuous, and not integrable on  $R_+$ .

**Example 6.** Consider the function

$$b(t) = \begin{cases} n & \text{if } t = n, n \geq 2, \\ \text{linear} & \text{if } t \in [n - 1/n^2, n] \cup [n, n + 1/n^2], n \geq 2, \\ 0 & \text{if } t \in [0, \frac{7}{4}] \cup [n + 1/n^2, (n + 1) - 1/(n + 1)^2], n \geq 2. \end{cases}$$

Then we have

$$\int_0^\infty b(s) ds = \sum_{n=2}^\infty \frac{1}{n} = \infty$$

and

$$\int_{n-1/n^2}^{n+1/n^2} b(s) ds = \frac{1}{2} \left( \frac{2}{n^2} \right) (n) = \frac{1}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

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