Quotient of Row Finite Matrix Semiring

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INTRODUCTION AND PRELIMINARIES

When A is a semiring and J is an ideal of A, the collection $\{x+J\}_{x\in A}$ of sets $x+J=\{x+j \mid j\in J\}$ need not be a partition of A. P. J. Allen [1] defined Q-ideal and maximal homomorphism and established the Fundamental Theorem of Homomorphisms in a large class of semirings.

Moreover, [2] builds the quotient structure in nxn matrix semirings. This paper is to prove an analogue of results for row finite matrix semirings.

The definitions of semiring, Q-ideal and maximal homomorphism used in [1] will be used throughout this paper. These definitions and theorems are given as follows.

Definition 1. A non-empty set A together with two associative binary operations called addition and multiplication (denoted by + and ·, respectively) will be called *a semiring* provided;

(1) addition is a commutative operation,

(2) there exists $0\in A$ such that $x+0=x$ and $x0=0x=0$ for all $x\in A$ and

(3) multiplication distributes over addition both from the left and from the right.

Definition 2. A non-empty subset J of a semiring A will be called *an ideal* if $a, b\in J$ and $r\in A$ implies $a+b\in J$, $ra\in J$ and $ar\in J$.

Definition 3. A mapping ϕ from the semiring A into the semiring A′ will be called *a homomorphism* if $\phi(a+b)=\phi(a)+\phi(b)$ and $\phi(ab)=\phi(a)\phi(b)$ for each $a, b\in A$. *An isomorphism* is an one-to-one homomorphism. The semirings A and A′ will be called *isomorphic* (denoted by $A\cong A'$) if there exists an isomorphism from A onto A′.

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Definition 4. An ideal J in the semiring A will be called *a Q-ideal* if there exists a subset Q of A satisfying the following conditions;

(1) $\{q+J\}_{q\in Q}$ is a partition of A and

(2) if $q_1, q_2 \in Q$ such that $q_1 \neq q_2$, then

$(q_1+J)\cap(q_2+J)=\phi$.

Definition 5. A homomorphism ϕ from the semiring A onto the semiring A′ is said to be *maximal* if for each $a\in A'$ there exists $c_a\in\phi^{-1}(\{a\})$ such that $x+\ker\phi\subset c_a+\ker\phi$ for each $x\in\phi^{-1}(\{a\})$, where $\ker\phi=\{x\in A\mid\phi(x)=O\}$.

Lemma 6. Let J be a Q-ideal in the semiring A. If $x\in A$, then there exists a unique $q\in Q$ such that $x+J\subset q+J$.

Theorem 7. If J is a Q-ideal in the semiring A, then $A/J=(\{q+J\}_{q\in Q}, \oplus_Q, \odot_Q)$ is a semiring.

Theorem 8. If ϕ is a maximal homomorphism from the semiring A onto the semiring A′, then $A/\ker\phi\cong A'$.

THE QUOTIENT OF ROW FINITE MATRIX SEMIRING

Consider a semiring A and nonempty countable index set I. Mappings $M: I\times I\to A$ are called *matrices over A*. The values of M are denoted by m_{ij}, where $i,j\in I$. The values m_{ij} are also referred to as *the entries* of the matrix. In particular, m_{ij} is called the *(i,j)-entry* of M. The matrix M is denoted by $[m_{ij}]$ and the collection of all matrices M over A as defined above is denoted by $[A]^{I\times I}$.

For each $M=[m_{ij}]\in[A]^{I\times I}$ and each $i\in I$, consider the set of indices $R_M(i)=\{j\in I\mid m_{ij}\neq O\}$. Then M is called *a row finite matrix* iff $R_M(i)$ is finite for all $i\in I$. The collection of all row finite matrices over A as defined above is denoted by $[A]^{I\times I}_{RF}$.

Theorem 9. If A is a semiring, then $[A]^{I\times I}_{RF}$ is also.

Proof. For $M=[m_{ij}]$, $N=[n_{ij}]\in[A]^{I\times I}_{RF}$, we define the addition and the multiplication by

$M+N=[m_{ij}+n_{ij}]\in$ for all $i,j\in I$ and

$MN=[\sum_{j\in I} m_{ij}n_{jk}]$ for all $i,k\in I$.

Then the addition and the multiplication are well-defined operations on $[A]^{I\times I}_{RF}$ since $R_{M+N}(i)\subset R_M(i)\cup R_N(i)$ for all $i\in I$,

$\sum_{j\in I} m_{ij}n_{jk}=\sum_{j\in R_M(i)} m_{ij}n_{jk}$ and

$R_{MN}(i)\subset\bigcup_{j\in R_M(i)} R_N(j)$

Now we introduce the zero matrix denoted by O that the entries of O are O. Then O is an additive zero.

Furthermore, the addition is commutative and associative and the multiplication is associative and distributes over addition both from the left and from the right. Hence $[A]^{I\times I}_{RF}$ is also a semiring.

Corollary 10. If A is a semiring and J is a Q-ideal of A, then $[A/J]^{I\times I}_{RF}$ is a semiring.

Proof. It is obvious by Theorem 7 and Theorem 9. In this corollary, the binary operations are defined as follows;

(1) $[q'_{ij}+J]+[q''_{ij}+J]=[q_{ij}+J]$

where $q'_{ij}+q''_{ij}+J\subset q_{ij}+J$ for all $i,j\in I$

(2) $[q'_{ij}+J][q''_{ij}+J]=[q_{ij}+J]$

where $\sum_{k\in I} q'_{ik}q''_{kj}+J\subset q_{ij}+J$ for all $i,j\in I$.

Since $M=[q'_{ij}+J]$ is row finite, the range of k in (2) is $R_M(i)$. So the range of k is finite.

Theorem 11. If A is a semiring and J is a Q-ideal in A, then $[J]^{I\times I}_{RF}$ is a $[Q]^{I\times I}_{RF}$-ideal in $[A]^{I\times I}_{RF}$.

Proof. It is clear that $[J]^{I\times I}_{RF}$ is an ideal in $[A]^{I\times I}_{RF}$.

(1) Suppose $[m_{ij}]\in[A]^{I\times I}_{RF}$. Since $m_{ij}\in A$ for all $i, j\in I$ and J is a Q-ideal in A, $m_{ij}\in \bigcup_{q\in Q}(q+J)$ for all $i,j\in I$. i.e. for all $i,j\in I$, $m_{ij}=q_{ij}+n_{ij}$ for some $q_{ij}\in Q$ and some $n_{ij}\in J$. Thus $[m_{ij}]=[q_{ij}+n_{ij}]=[q_{ij}]+[n_{ij}]\in P+[J]^{I\times I}_{RF}$ for some $P=[q_{ij}]\in[Q]^{I\times I}_{RF}$.

Hence $[m_{ij}]\in \bigcup_{P\in[Q]^{I\times I}_{RF}}\{P+[J]^{I\times I}_{RF}\}$.

(2) Let $[p_{ij}]$ and $[q_{ij}]$ be in $[Q]^{I\times I}_{RF}$ and let $[p_{ij}]\neq[q_{ij}]$. Then there exist $i,j\in I$ such that $p_{ij}\neq q_{ij}$.

Since J is a Q-ideal in A, $(p_{ij}+J)\cap(q_{ij}+J)=\phi$.

So, $p_{ij}+m\neq q_{ij}+n$ for all $m,n\in J$.

Consequently, the (i,j)-entry of every matrices in $[p_{ij}]+[J]^{I\times I}_{RF}$ is different from the (i,j)-entry of every matrices in $[q_{ij}]+[J]^{I\times I}_{RF}$.

Thus $([p_{ij}]+[J]^{I\times I}_{RF})\cap([q_{ij}]+[J]^{I\times I}_{RF})=\phi$.

Hence $[J]^{I\times I}_{RF}$ is a $[Q]^{I\times I}_{RF}$-ideal in $[A]^{I\times I}_{RF}$.

Corollary 12. If A is a semiring and J is a Q-ideal in A, then $[A]^{I\times I}_{RF}/[J]^{I\times I}_{RF}=(\{P+[J]^{I\times I}_{RF} \mid P\in[Q]^{I\times I}_{RF}\}, +_{[Q]^{I\times I}_{RF}}, \cdot_{[Q]^{I\times I}_{RF}})$ is a semiring.

Proof. This corollary is the immediate result of Theorem 11 and Theorem 7.

The operations are as follows;

(1) $(P_1+[J]^{I\times I}_{RF})+_{[Q]^{I\times I}_{RF}}(P_2+[J]^{I\times I}_{RF})=P+[J]^{I\times I}_{RF}$

where $P_1+P_2+[J]^{I\times I}_{RF}\subset P+[J]^{I\times I}_{RF}$ and

(2) $(P_1+[J]^{I\times I}_{RF})\cdot_{[Q]^{I\times I}_{RF}}(P_2+[J]^{I\times I}_{RF})=P+[J]^{I\times I}_{RF}$ where $P_1P_2+[J]^{I\times I}_{RF}\subset P+[J]^{I\times I}_{RF}$

Theorem 13. If A is a semiring and J is a Q-ideal in A, then $[A]^{I\times I}_{RF}/[J]^{I\times I}_{RF}$ is isomorphic to $[A/J]^{I\times I}_{RF}$.

Proof. For each $m_{ij}\in A$, there exists a unique $q_{ij}\in Q$ such that $m_{ij}+J\subset q_{ij}+J$ by Lemma 6. Define the map $\phi:[A]^{I\times I}_{RF}\to[A/J]^{I\times I}_{RF}$ by $\phi([m_{ij}])=[q_{ij}+J]$ for each $[m_{ij}]\in[A]^{I\times I}_{RF}$, where $m_{ij}+J\subset q_{ij}+J$ for each $i, j\in I$. Then it is clear that ϕ is a homomorphism from $[A]^{I\times I}_{RF}$ onto $[A/J]^{I\times I}_{RF}$ and $\ker\phi=[J]^{I\times I}_{RF}$ by proposition 14 in [2]. For each $[q_{ij}+J]\in[A/J]^{I\times I}_{RF}$, $[q_{ij}]\in\phi^{-1}([q_{ij}+J])$. If $[a_{ij}]\in\phi^{-1}([q_{ij}+J])$, then $a_{ij}+J\subset q_{ij}+J$ for all $i, j\in I$. Thus $[a_{ij}]+\ker\phi=[a_{ij}]+[J]^{I\times I}_{RF}\subset[q_{ij}]+[J]^{I\times I}_{RF}=[q_{ij}]+\ker\phi$. Hence ϕ is a maximal homomorphism from the semiring $[A]^{I\times I}_{RF}$ onto the semiring $[A/J]^{I\times I}_{RF}$.

Therefore $[A]^{I\times I}_{RF}/[J]^{I\times I}_{RF}\sim[A/J]^{I\times I}_{RF}$ by Theorem 8.

LITERATURES CITED

[1] Allen, P. J. 1969; *A fundamental theorem of homomorphism for semirings*, Proc. Amer. Math. Soc. 21; 412-416

[2] Yang, S. 1983; *Quotients of Matrix Semiring*, Cheju University J. 15, Natural Sciences, 133-135.

國 文 抄 錄

이 논문에서는 A가 semiring이고 J가 A에서의 Q- ideal이면 $[J]_{RF}^{I\times I}$는 $[A]_{RF}^{I\times I}$에서 $[Q]_{RF}^{I\times I}$ - ideal이 되어 $[A]_{RF}^{I\times I}/[J]_{RF}^{I\times I}$는 semiring이 됨을 보였고 또 $[A]_{RF}^{I\times I}/[J]_{RF}^{I\times I}$와 $[A/J]_{RF}^{I\times I}$는 서로 동형임을 보였다.