FUZZY DUAL IDEALS IN BCK-ALGEBRAS (III)

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ABSTRACT. We consider completely normalized fuzzy dual ideals in BCK-algebras.

In developing an algebraic theory of BCK-algebras, the notion of ideals has played an important role. In [7] - [13] J. Meng introduced the concept of dual ideals in BCK-algebras and obtained a number of its properties. The concept of fuzzy sets was introduced by Zadeh [16]. Xi [15] applied the concept of fuzzy set to BCK-algebra, and he got some results. In [14], Meng and Jun introduced the notion of fuzzy dual ideals in BCK-algebras, and investigated some interesting results. The present paper is a continuation of [5], [6] and [14]. In this paper we investigate further properties of fuzzy dual ideals and normalized fuzzy dual ideals. We also introduce the notion of completely normalized fuzzy dual ideals in BCK-algebras, and study their properties.

By a BCK-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

for all $x, y, z \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra X is said to be bounded if there exists an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. We will denote $1 * x$ by Nx for brief. We note that $N1 = 0$ and $N0 = 1$ in a bounded BCK-algebra. In what follows, X would mean a bounded BCK-algebra unless otherwise specied.

Definition 1 ([16]). A fuzzy set μ in a set X is a function $\mu : X \to [0, 1]$.

Definition 2 ([16]). Let μ be a fuzzy set in a set X. For $t \in [0, 1]$, the set

$$\mu_t = \{x \in X | \mu(x) \geq t\}$$

is called a level subset of μ.

Definition 3 ([7]). A nonempty subset D of X is called a dual ideal of X if it satisfies:
(i) $1 \in D$,
(ii) $N(Nx * Ny) \in D$ and $y \in D$ imply $x \in D$.

We shall write $a \wedge b$ for $\min\{a, b\}$, where a and b are any two real numbers.

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Definition 4 ([14]). A fuzzy set μ in X is called a fuzzy dual ideal of X if it satisfies:
(i) $\mu(1) \geq \mu(x)$ for all x in X,
(ii) $\mu(x) \geq \mu(N(Nx * Ny)) \wedge \mu(y)$ for all x, y in X.

Example 1 ([5]). Let $X = \{0, a, b, 1\}$. The operation $*$ is defined by

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then X is a bounded BCK-algebra. Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = \mu(a) = t_1$, $\mu(b) = t_2$ and $\mu(1) = t_3$ where $0 \leq t_1 < t_2 < t_3 \leq 1$. A straightforward computation shows that μ is a fuzzy dual ideal of X.

Proposition 1. *If μ is a fuzzy dual ideal of X, then $\mu(0) \leq \mu(x)$ for all $x \in X$.*

Proof. Taking $y = 0$ in Definition 4(ii) and using Definition 4(i) we have

$$\mu(x) \geq \mu(N(Nx * N0)) \wedge \mu(0) = \mu(1) \wedge \mu(0) = \mu(0).$$

Let T be a nonempty subset of $[0, 1]$.

Theorem 1. *Let $\{D_t | t \in T\}$ be a collection of dual ideals of X such that $X = \cup D_t$ and for all $s, t \in T$, $s > t$ if and only if $D_s \subset D_t$. Define the fuzzy set μ in X by*

$$\mu(x) = \sup\{t | x \in D_t\} \text{ for all } x \in X.$$

Then μ is a fuzzy dual ideal of X.

Proof. Since $1 \in D_t$ for all $t \in T$, clearly $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $\mu(N(Nx * Ny)) = m$ and $\mu(y) = n$. Without loss of generality we may assume that $m \leq n$. To prove $\mu(x) \geq \mu(N(Nx * Ny)) \wedge \mu(y)$, we consider the following three cases:

$$\text{(i) } t \leq m, \text{ (ii) } m < t \leq n \text{ and (iii) } t > n.$$

Case (i) implies that $N(Nx * Ny) \in D_t$ and $y \in D_t$. Since D_t is a dual ideal of X, it follows that $x \in D_t$, so that

$$\mu(x) = \sup\{t | x \in D_t\} = m = \mu(N(Nx * Ny)) \wedge \mu(y).$$

For the case (ii), we have $N(Nx * Ny) \notin D_t$ and $y \in D_t$. It follows that either $x \in D_t$ or $x \notin D_t$. If $x \in D_t$, then

$$\mu(x) = n \geq \mu(N(Nx * Ny)) \wedge \mu(y).$$

If $x \notin D_t$, then $x \in D_k - D_t$ for all $k < t$, and so $\mu(x) \geq m = \mu(N(Nx * Ny)) \wedge \mu(y)$. Finally case (iii) implies $N(Nx * Ny) \notin D_t$ and $y \notin D_t$. Then we also have either $x \in D_t$ or $x \notin D_t$. If $x \in D_t$, then obviously $\mu(x) \geq \mu(N(Nx * Ny)) \wedge \mu(y)$. If $x \notin D_t$, then $x \in D_r - D_t$ for all $r < t$, and thus $\mu(x) \geq m = \mu(N(Nx * Ny)) \wedge \mu(y)$. This completes the proof.

Lemma 1 ([5]). *If μ is a fuzzy dual ideal of X, then the set $X_\mu = \{x \in X | \mu(x) = \mu(1)\}$ is a dual ideal of X.*

We now define a construction of new fuzzy dual ideals from old. Let $t \geq 0$ be a real number. If $m \in [0,1]$, m^t shall mean the positive root in case $t < 1$. We define $\mu^t : X \to [0,1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in X$.

Theorem 2. *If μ is a fuzzy dual ideal of X, then μ^t is also a fuzzy dual ideal of X and $X_{\mu^t} = X_\mu$.*

Proof. Let $x, y \in X$. Then $\mu^t(1) = (\mu(1))^t \geq (\mu(x))^t = \mu^t(x)$ and

$$\begin{aligned}\mu^t(x) &= (\mu(x))^t \\ &\geq (\mu(N(Nx * Ny)) \wedge \mu(y))^t \\ &= (\mu(N(Nx * Ny)))^t \wedge (\mu(y))^t \\ &= \mu^t(N(Nx * Ny)) \wedge \mu^t(y).\end{aligned}$$

Hence μ^t is a fuzzy dual ideal of X. Now

$$\begin{aligned}X_{\mu^t} &= \{x \in X | \mu^t(x) = \mu^t(1)\} \\ &= \{x \in X | (\mu(x))^t = (\mu(1))^t\} \\ &= \{x \in X | \mu(x) = \mu(1)\} \\ &= X_\mu\end{aligned}$$

Definition 5 ([5]). A fuzzy set μ in X is called a normalized fuzzy dual ideal of X if μ is a fuzzy dual ideal of X satisfying $\mu(1) = 1$.

Example 2. Let μ be as in Example 1. If we take $t_3 = 1$ in Example 1, then μ is a normalized fuzzy dual ideal of X.

Lemma 2 ([5]). *Let D be a dual ideal of X. Define a function $\mu_D : X \to [0,1]$ by*

$$\mu_D(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{if } x \notin D. \end{cases}$$

Then μ_D is a normalized fuzzy dual ideal of X and $X_{\mu_D} = D$.

For any fuzzy sets μ and ν in X, we define

$$\mu \subseteq \nu \Leftrightarrow \mu(x) \leq \nu(x) \text{ for all } x \in X.$$
$$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x) \text{ for all } x \in X.$$

The following is easily verified.

Proposition 2. *If μ and ν are normalized fuzzy dual ideals of X then so is $\mu \cap \nu$.*

Lemma 3 ([5]). *Let μ be a fuzzy dual ideal of X and let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(1)$ for all $x \in X$. Then μ^+ is a normalized fuzzy dual ideal of X and $\mu \subseteq \mu^+$.*

Let $\mathcal{D}(X)$ (resp. $\mathcal{NFD}(X)$) denote the set of all dual (resp. normalized fuzzy dual) ideals of X. We can define functions $\phi : \mathcal{D}(X) \to \mathcal{NFD}(X)$ and $\psi : \mathcal{NFD}(X) \to \mathcal{D}(X)$ by $\phi(D) = \mu_D$ and $\psi(\mu) = X_\mu$. Then $\psi\phi = 1_{\mathcal{D}(X)}$ and $\phi\psi(\mu) = \phi(X_\mu) = \mu_{X_\mu} \subseteq \mu$.

Theorem 3. *If D and E are dual ideals of X, then $\mu_{D\cap E} = \mu_D \cap \mu_E$. If μ and ν are normalized fuzzy dual ideals of X, then $X_{\mu\cap\nu} = X_\mu \cap X_\nu$. Thus $\phi(D \cap E) = \phi(D) \cap \phi(E)$ and $\psi(\mu \cap \nu) = \psi(\mu) \cap \psi(\nu)$.*

Proof. Let $x \in X$. If $x \in D \cap E$, then $\mu_{D\cap E}(x) = 1$. From $x \in D$ and $x \in E$ it follows that $\mu_D(x) = 1 = \mu_E(x)$. Hence $\mu_{D\cap E}(x) = 1 = \mu_D(x) \wedge \mu_E(x) = (\mu_D \cap \mu_E)(x)$. If $x \notin D \cap E$ then $x \notin D$ or $x \notin E$. Thus $\mu_{D\cap E}(x) = 0 = \mu_D(x) \wedge \mu_E(x) = (\mu_D \cap \mu_E)(x)$. Therefore $\mu_{D\cap E} = \mu_D \cap \mu_E$. Let $\mu, \nu \in \mathcal{NFD}(X)$. Then

$$\begin{aligned} X_{\mu\cap\nu} &= \{x \in X | (\mu \cap \nu)(x) = (\mu \cap \nu)(1)\} \\ &= \{x \in X | \mu(x) \wedge \nu(x) = 1\} \\ &= \{x \in X | \mu(x) = 1 \text{ and } \nu(x) = 1\} \\ &= \{x \in X | \mu(x) = 1\} \cap \{x \in X | \nu(x) = 1\} \\ &= \{x \in X | \mu(x) = \mu(1)\} \cap \{x \in X | \nu(x) = \nu(1)\} \\ &= X_\mu \cap X_\nu. \end{aligned}$$

Thus

$$\phi(D \cap E) = \mu_{D\cap E} = \mu_D \cap \mu_E = \phi(D) \cap \phi(E)$$

and

$$\psi(\mu \cap \nu) = X_{\mu\cap\nu} = X_\mu \cap X_\nu = \psi(\mu) \cap \psi(\nu).$$

This completes the proof.

Definition 6 ([5]). A fuzzy dual ideal μ of X is called a fuzzy maximal dual ideal if
 (i) μ is non-constant,
 (ii) μ^+ is a maximal element of $(\mathcal{NFD}(X), \subseteq)$.

Lemma 4 ([5]). *Let $\mu \in \mathcal{NFD}(X)$ be non-constant such that it is a maximal element of $(\mathcal{NFD}(X), \subseteq)$. Then $Im(\mu) = \{0, 1\}$.*

Lemma 5 ([5]). *If μ is a fuzzy maximal dual ideal of X, then*
 (i) *μ is normalized,*
 (ii) *$Im(\mu) = \{0, 1\}$,*
 (iii) *$\mu_{X_\mu} = \mu$,*
 (iv) *X_μ is a maximal dual ideal of X.*

Definition 7. A fuzzy set μ in X is called a completely normalized fuzzy dual ideal of X if it satisfies:
 (i) μ is a normalized fuzzy dual ideal of X.
 (ii) $\mu(0) = 0$.

Example 3. Let μ be as in Example 1. If we take $t_1 = 0$ and $t_3 = 1$ in Example 1, then μ is a completely normalized fuzzy dual ideal of X.

Theorem 4. *Assume that μ is a fuzzy dual ideal of X and let μ^- be a fuzzy set in X defined as follows:*

$$\mu^-(x) = (\mu(x) - \mu(0))/(\mu(1) - \mu(0)) \text{ for all } x \in X.$$

Then μ^- is a completely normalized fuzzy dual ideal of X.

Proof. Following Proposition 1, we know that μ^- is well-defined. Clearly $\mu^-(0) = 0$ and $\mu^-(1) = 1 \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned}
&\mu^-(N(Nx * Ny)) \wedge \mu^-(y) \\
&= ((\mu(N(Nx * Ny)) - \mu(0))/(\mu(1) - \mu(0))) \wedge ((\mu(y) - \mu(0))/(\mu(1) - \mu(0))) \\
&= (1/(\mu(1) - \mu(0)))((\mu(N(Nx * Ny)) - \mu(0)) \wedge (\mu(y) - \mu(0))) \\
&= (1/(\mu(1) - \mu(0)))((\mu(N(Nx * Ny)) \wedge \mu(y)) - \mu(0)) \\
&\leq (\mu(x) - \mu(0))/(\mu(1) - \mu(0)) \\
&= \mu^-(x),
\end{aligned}$$

proving the theorem.

The following theorem is obvious, and we omit the proof.

Theorem 5. *If μ is a completely normalized fuzzy dual ideal of X, then $\mu^- = \mu$.*

Let $\mathcal{CNFD}(X)$ denote the set of all completely normalized fuzzy dual ideals of X. Clearly $\mathcal{CNFD}(X) \subset \mathcal{NFD}(X)$. The restriction of the partial ordering $\subseteq$ of $\mathcal{NFD}(X)$ gives a partial ordering of $\mathcal{CNFD}(X)$.

Theorem 6. *Every non-constant maximal element of $(\mathcal{NFD}(X), \subseteq)$ is also a maximal element of $(\mathcal{CNFD}(X), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{NFD}(X), \subseteq)$. By Lemma 4, $Im(\mu) = \{0, 1\}$, and in fact $\mu(1) = 1$ and $\mu(0) = 0$. Hence $\mu \in \mathcal{CNFD}(X)$. Assume that there exists $\nu \in \mathcal{CNFD}(X)$ such that $\mu \subseteq \nu$. It follows that $\mu \subseteq \nu$ in $(\mathcal{NFD}(X), \subseteq)$. Since μ is amximal in $(\mathcal{NFD}(X), \subseteq)$ and since ν is non-constant, therefore $\mu = \nu$. Thus μ is a maximal element of $(\mathcal{CNFD}(X), \subseteq)$.

Theorem 7. *Every fuzzy maximal dual ideal of X is completely normalized.*

Proof. Let μ be a fuzzy maximal dual ideal of X. Then by Lemma 5, μ is normalized and $\mu = \mu^+$ takes only the values $\{0, 1\}$. Now we claim that $\mu(0) = 0$. If not, then $\mu(0) = 1$. Hence $\mu(x) \geq \mu(0) = 1$, and hence $\mu(x) = 1$ for all $x \in X$. This means that μ is constant, which is a contradiction. The proof is complete.

The following theorem is obvious.

Theorem 8. *If μ is a completely normalized fuzzy dual ideal of X, then so is μ^t for a real number $t \geq 0$.*

Theorem 9. *For every maximal element μ of $\mathcal{CNFD}(X)$ we have $Im(\mu) = \{0, 1\}$ and $\mu = \mu_{X_\mu}$.*

proof. Assume that there exists $x_0 \in X$ such that $0 < \mu(x_0) < 1$. Then we can take $0 < t < 1$ and consider μ^t. By Theorem 8 we get $\mu^t \in \mathcal{CNFD}(X)$ and clearly $\mu(x) \leq \mu^t(x)$ for all $x \in X$. It follows, in particular, that $\mu(x_0) < \mu^t(x_0)$, which contradicts the maximality of μ. Next obviously $\mu_{X_\mu} \leq \mu$, and $Im(\mu_{X_\mu}) = \{0, 1\}$. Let $x \in X$. If $\mu(x) = 0$, then clearly $\mu \leq \mu_{X_\mu}$. If $\mu(x) = 1$ then $x \in X_\mu$, and so $\mu_{X_\mu}(x) = 1$. This proves that $\mu \leq \mu_{X_\mu}$ and the proof is complete.

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