Bull. Korean Math. Soc. 33(1996), No. 2, pp. 427-434

# SPECTRA AND NUMERICAL RANGES

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## 1. Introduction

The theory of numerical ranges in unital normed algebras has extensively studied by many authors, for example, see [1], [2] for details.

Let $x$ be a fixed element of a non unital normed algebra $A$ over a field $K = \mathbb{R}$ or $\mathbb{C}$. In [8], Yang introduced the notion of right(left) relative numerical range $V_x^R(A,a)(V_x^L(A,a))$ of an element $a$ of a non unital normed algebra $A$ relative to $x \in A$(see Definition 2.1). If $x = e$, the identity of $A$ and $\|e\| = 1$, then $V_x^R(A,a)$ coincides with $V(a)$ where $V(a)$ denotes the (Bonsall and Duncan) numerical range of $a$ ([2]). Since $x$ can be arbitrarily chosen, this concept extends the familiar concept of numerical range. Among the results, it is shown ([8]) that our numerical range is a compact convex subset of $K$.

In this paper, we introduce the notion of w-proximinal subspace of a normed linear space and show that every proper closed left or right ideal of a unital Banach algebra is w-proximinal. Also we give an example which is not a w-proximinal subspace.

It is shown ([1]) that if $A$ is a unital Banach algebra, then $Sp_A(a) \subset V(a)$. Further, we introduce the notions of proximinal properties of a normed algebra, and for a normed algebra $A$ having proximinal property, we give the similar inclusion relation between the spectrum $Sp_A(a)$ and the relative numerical range of $a \in A$. In particular we show that if $A$ is a normed algebra having the first proximinal property or second proximinal property and $a \in A$, then $\{0\}^c \cap Sp_A(a) \subset V_x(A,a)$ for some $x \in A$ with $\|x\| = 1$.

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Received December 9, 1995. Revised March 11, 1996.

1991 AMS Subject Classification: 47A12.

Key words and phrases: relative numerical ranges, proximinal.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994.

## 2. Spectra and relative numerical ranges

In this section, we introduce the concept of proximinal property of a normed algebra and study the inclusion relation between the spectrum and the relative numerical range.

DEFINITION 2.1. ([8]) Let $A$ be a normed algebra over the field $K = \mathbb{R}$ or $\mathbb{C}$, and $A'$ its dual. For $x \in A$, we write

$$D(A, x) = \{f \in A' : \|f\| = 1, f(x) = \|x\|\}.$$

The right relative numerical range of $a \in A$ relative to $x$ is defined to be $V_x^R(A, a) = \{f(ax) : f \in D(A, x)\}$. The left relative numerical range of $a \in A$ relative to $x$ is defined to be $V_x^L(A, a) = \{f(xa) : f \in D(A, x)\}$. The relative numerical range of $a$ relative to $x$ is defined to be $V_x(A, a) = V_x^R(A, a) \cup V_x^L(A, a)$. The right relative numerical radius of $a$ relative to $x$ is defined by $v_x^R(a) = \sup\{|\lambda| : \lambda \in V_x^R(A, a)\}$. The left relative numerical radius of $a$ relative to $x$ is defined by $v_x^L(a) = \sup\{|\lambda| : \lambda \in V_x^L(A, a)\}$. The relative numerical radius of $a$ relative to $x$ is defined by $v_x(a) = \max\{v_x^R(a), v_x^L(a)\}$.

Note that the set $D(A, x)$ is a nonempty subset of $A'$ by the Hahn-Banach Theorem, and so $V_x^R(A, a)$ and $V_x^L(A, a)$ are nonempty. If $A$ is commutative, then $V_x^R(A, a) = V_x^L(A, a) = V_x(A, a)$ as $f(ax) = f(xa)$. If $x = e$(identity of $A$) with $\|e\| = 1$, then $V_e(A, a) = V(a)$, where $V(a)$ denotes the (Bonsall and Duncan) numerical range of $a$ ([2]). Thus the concept of relative numerical range extends that of the (Bonsall and Duncan) numerical range.

LEMMA 2.2. ([8]) *Let $a, x$ be elements of a normed algebra $A$. Then*

(1) *$D(A, x)$ is a weak* compact convex subset of $A'$.*
(2) *$V_x^R(A, a)$ and $V_x^L(A, a)$ are compact convex subsets of $K$, hence $V_x(A, a)$ is a compact subset of $K$.*
(3) *If $B$ is a subalgebra of a normed algebra $A$ and $b, x \in B$, then $V_x(B, b) = V_x(A, b)$.*

We introduce the concept of a w-proximinal subspace in the following:

DEFINITION 2.3. Let $X$ be a normed linear space and $M$ a proper subspace of $X$. $M$ is called w-proximinal if there exists an element $z \in X$ such that $1 = \|z\| = d(z, M)$.

It is easy to show that every proximal subspace of a normed space is w-proximinal. But it is unknown whether the converse holds or not.

EXAMPLE 2.4. (1) Let $X = \mathbb{R}^2$ and let $M$ be the $x$-axis. Then $M$ is clearly a proper subspace of a normed linear space $X$, and $1 = \|x\| = d(x, M)$ for some $x = (0, 1) \in X$.

(2) Let $X = l^\infty$ be the set of all bounded sequences and let $M = c_0$ be the set of all sequences that converge to 0. Then $M$ is a closed subspace of $X$ and hence is a Banach space(not reflexive). Also $1 = \|x\| = d(x, c_0)$ for some $x = \{1 - \frac{1}{n}\}$ in $X$. Hence $c_0$ is a w-proximinal subspace of $l^\infty$.

LEMMA 2.5. ([4]) *Suppose $X$ is a normed linear space over $\mathbb{R}$ or $\mathbb{C}$. Let $M$ be a finite dimensional proper subspace of $X$. Then there exists an $x \in X$ such that $1 = \|x\| = d(x, M)$ where $d(x, M)$ is the distance from $x$ to $M$.*

From the above lemma, every finite dimensional proper subspace of a normed linear space is w-proximinal.

EXAMPLE 2.6. Let $L : C[0,1] \longrightarrow K$ be defined by

$$L(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^1 f(x)dx.$$

Then $\|L\| = 1$ and $d(f, \ker L) = |L(f)|$ by ([7], problem 3, p138). Thus there does not exist $g \in C[0,1]$ such that $1 = \|g\| = |L(g)|$, i.e., $\ker L$ is not w-proximinal.

Let $A$ be a normed algebra but do not assume that $A$ has an identity. We say that a left ideal $I$ is modular if there exists $e \in A$ such that $A(1-e) \equiv \{a - ae : a \in A\} \subset I$. $e$ is called a right modular unit for $I$. Similarly, a right ideal $I$ is modular if there exists a left modular unit for $I$. A two-sided ideal $I$ is modular if it is modular both as a left and as a right ideal([3]).

DEFINITION 2.7. Let $A$ be a normed algebra. $A$ is said to have the first proximinal property if every one-sided maximal modular ideal of $A$ is w-proximinal. $A$ is said to have the second proximinal property if every maximal modular ideal of a maximal commutative subalgebra of $A$ is w-proximinal.

By definition, a w-proximinal subspace of a normed linear space has proper closure. So every w-proximinal maximal modular left ideal is closed.

LEMMA 2.8. ([5]) *Let $M$ be a subspace of a normed linear space $X$. Given $x \in X$ with $d = d(x, M) > 0$, there exists an $f \in X'$ such that*

$$\|f\| = 1, \quad f(M) = \{0\} \text{ and } f(x) = d(x, M).$$

Let $r(A)$ denotes the set of regular elements in a normed algebra $A$ and $N(a, r)$ the open ball at $a \in A$ of radius $r$. For an element $a$ of $A$, let $Sp_A(a)$, $Sp_A^L(a)\,(Sp_A^R(a))$ denote the spectrum, left( right) spectrum of $a$ with respect to $A$ respectively.

THEOREM 2.9. *Let $A$ be a unital Banach algebra. Then every proper closed left or right ideal $M$ of $A$ is w-proximinal. In particular, any unital Banach algebra has the first and second proximinal properties.*

*Proof.* Let $M$ be any proper closed left or right ideal of $A$. If $x \in N(e, 1)$, then $\|e - x\| < 1$ and so $x$ is regular in $A$ by ([3], Lemma 2.1). Hence $e \in N(e, 1) \subseteq r(A) \subseteq M^c$ and $e \notin M$. Put $r = d(e, M)$. By Lemma 2.8, there exists $g \in A'$ such that $g(e) = r$, $\|g\| = 1$ and $g(M) = \{0\}$. Since $e$ is an identity of $A$, $g(e) = 1$ and $\|e\| = 1$. Hence $1 = g(e) = r = d(e, M)$, i.e., $M$ is w-proximinal.

It is shown ([1]) that if $A$ is a unital Banach algebra, then $Sp_A(a) \subset V(a)$. The following theorem gives the similar inclusion relation between the spectrum $Sp_A(a)$ and the relative numerical range of $a$ in a non unital normed algebra $A$.

THEOREM 2.10. *Let $A$ be a normed algebra over $\mathbb{C}$ having the first or second proximinal property, and $a \in A$. Then $\{0\}^c \cap Sp_A^L(a) \subset V_x^L(A, a)$ for some $x \in A$ with $\|x\| = 1$.*

*Proof.* Assume that $A$ has the first proximinal property. Let $\lambda \in Sp_A^L(a)$ and $\lambda \neq 0$. Define $j \equiv \lambda^{-1}a$. Then $j$ is not left quasi-regular, i.e. there does not exist a $h \in A$ such that $h \circ j \equiv h + j - hj = 0$. Thus $A(1-j)$ is a left modular ideal $M$ of $A$, so $A(1-j)$ is contained in a left maximal modular ideal of $A$. By the first proximinal property, there exists $x \in A$ with $1 = \|x\| = d(x, M)$. By Lemma 2.8, there exists $f \in A^*$ such that $f(M) = \{0\}, 1 = \|f\| = f(x)$. Also $x - xj \in M$ implies $0 = f(x - xj) = f(x) - f(xj) = 1 - \lambda^{-1}f(xa)$. Then $f(xa) = \lambda \in V_x^L(A, a)$

Assume that $A$ has the second proximinal property. For $a \in A$, $a$ is an element of a maximal commutative subalgebra $B$ of $A$. Let $\lambda \in \{0\}^c \cap Sp_A(a) = \{0\}^c \cap Sp_B(a)$. The second proximinal property for $A$ implies the first proximinal property for $B$, so by the first argument, $\lambda \in V_x^L(B, a) \subseteq V_x^L(A, a)$ for some $x \in B$.

THEOREM 2.11. *Let $A$ be a normed algebra over $\mathbb{C}$ having the first or second proximinal property and $a \in A$. Then $\{0\}^c \cap Sp_A^R(a) \subset V_x^R(A, a)$ for some $x \in A$ with $\|x\| = 1$.*

*Proof.* The proof is similar to that of Theorem 2.10.

COROLLARY 2.12. *Let $A$ be a normed algebra over $\mathbb{C}$ having the first or second proximinal property and $a \in A$. Then $\{0\}^c \cap Sp_A(a) \subset V_x(A, a)$ for some $x \in A$ with $\|x\| = 1$.*

*Proof.* Since $Sp_A(a) = Sp_A^L(a) \cup Sp_A^R(a)$, for any $\lambda \in \{0\}^c \cap Sp_A(a)$, either $\lambda \in \{0\}^c \cap Sp_A^L(a)$ or $\lambda \in \{0\}^c \cap Sp_A^R(a)$. Assume that $\lambda \in \{0\}^c \cap Sp_A^L(a)$. Then by Theorem 2.10,

$$\lambda \in V_x^L(A, a) \subseteq V_x(A, a) \quad \text{for some} \quad x \in A.$$

Similarly if $\lambda \in \{0\}^c \cap Sp_A^R(a)$, then by Theorem 2.11,

$$\lambda \in V_z^R(A, a) \subseteq V_z(A, a) \quad \text{for some} \quad z \in A.$$

Therefore $\{0\}^c \cap Sp_A(a) \subset V_x(A, a)$ for some $x \in A$.

THEOREM 2.13. *Let $A$ be a normed algebra over $\mathbb{C}$ with the first or second proximinal property and an identity $e$, but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A^L(a)$, then $0 \in V_z^L(A,a)$ for some $z \in A$ with $\|z\| = 1$.*

*Proof.* If $A$ has the first proximinal property and $0 \in Sp_A^L(a)$, then $a$ is not left regular, so $Aa$ is a left modular ideal of $A$. Therefore $Aa$ is contained in a left maximal modular ideal $M$ of $A$. Note that since $A$ has $e$, all ideals are modular. By hypothesis, there exists $z \in A$ with $1 = \|z\| = d(z, M)$. Also by Lemma 2.8, there exists $g \in A'$ with $1 = \|g\| = g(z)$ and $g(M) = \{0\}$. Thus $za \in M$, so $0 = g(za) \in V_z^L(A,a)$.

If $A$ has the second proximinal property, then $a$ is an element of a maximal commutative subalgebra $B$ of $A$, and $e \in B$. Since $0 \in Sp_A^L(a) = Sp_B^L(a)$ and the second proximinal property for $A$ implies the first proximinal property for $B$, by the first argument, $0 \in V_z^L(B,a) = V_z^L(A,a)$ for some $z \in B$.

THEOREM 2.14. *Let $A$ be a normed algebra over $\mathbb{C}$ with the first or second proximinal property and an identity $e$, but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A^R(a)$, then $0 \in V_z^R(A,a)$ for some $z \in A$ with $\|z\| = 1$.*

*Proof.* The proof is similar to that of Theorem 2.13.

COROLLARY 2.15. *Let $A$ be a normed algebra over $\mathbb{C}$ with the first or second proximinal property and an identity $e$, but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A(a)$, then $0 \in V_z(A,a)$ for some $z \in A$ with $\|z\| = 1$.*

THEOREM 2.16. *Let $A$ be a normed algebra such that all proper closed one-sided ideals of $A$ are w-proximinal. Let $a \in A$ with $0 \in Sp_A(a)$. If $\overline{aA}$ or $\overline{Aa}$ is properly contained in $A$, then $0 \in V_z(A,a)$ for some $z \in A$ with $\|z\| = 1$.*

*Proof.* Clearly $\overline{aA}$ and $\overline{Aa}$ are proper closed one-sided ideals of $A$. By hypothesis there exists $z \in A$ with $1 = \|z\| = d(z, \overline{Aa})$ or $d(z, \overline{aA})$. By Lemma 2.8, there exists $g \in A'$ such that $g(z) = \|g\| = 1$ and $g(\overline{aA})$ or $g(\overline{Aa}) = \{0\}$. So $0 = g(za) \in V_z^L(A,a)$ or $0 = g(az) \in V_z^R(A,a)$. Hence $0 \in V_z(A,a)$

THEOREM 2.17. *Let $A$ be a finite dimensional complex normed algebra and let $a \in A$. Then*

(1) $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ *for some* $z \in A$ *with* $\|z\| = 1$.
(2) $aA$ *and* $Aa$ *are closed. If one of them is proper, then* $0 \in V_z(A, a)$ *for some* $z \in A$ *with* $\|z\| = 1$.
(3) *If* $aA = A = Aa$, *then* $A$ *has an identity* $e$, $a$ *is invertible and* $0 \notin Sp_A(a)$.
(4) $Sp_A(a) \subseteq V_z(A, a)$ *for some* $z \in A$ *with* $\|z\| = 1$.

*Proof.* By Lemma 2.4, all proper ideals are closed and w-proximinal.

(1) By Corollary 2.12, $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

(2) $aA$ and $Aa$ are clearly closed. If one of them is proper, then $a$ is singular i.e., $0 \in Sp_A(a)$. By Theorem 2.16, $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

(3) There are $z, w \in A$ such that $za = a = aw$. Also $x \in A$ implies $x = pa = aq$ for some $p, q \in A$. So $xw = paw = pa = x$ and $zx = zaq = aq = x$. Hence $e = z = w$ and $a$ is invertible.

(4) $(Aa) \cap (aA) \subset A$ or $= A$. If $(Aa) \cap (aA)$ is properly contained in $A$, then one of them is proper. By (1) and (2), $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ and $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$. Since $0 \in Sp_A(a)$, $Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

If $(Aa) \cap (aA) = A$, then by (3) $0 \notin Sp_A(a)$ and by (1) $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$. Hence $Sp_A(a) \subseteq V_z(A, a)$.

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