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# A COMPARISON OF MAXIMAL COLUMN RANKS OF MATRICES OVER RELATED SEMIRINGS

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## 1. Introduction

Let **A** be a real $m \times n$ matrix. The column rank of **A** is the dimension of the column space of **A** and the maximal column rank of **A** is defined as the maximal number of linearly independent columns of **A**. It is well known that the column rank is the maximal column rank in this situation.

However, we can also consider matrices whose entries come from another kind of algebraic system, such as a semiring or Boolean algebra. In this different context, the notions of column rank and maximal column rank can still be defined, but the two ranks do not necessarily agree. Indeed, Hwang, Kim and Song [6] compared the column rank and the maximal column rank for matrices over various semirings and found that except for small values of m and n, the two ranks did not agree in general.

In this paper we continue the study of maximal column rank, but instead of fixing the algebraic system and comparing the two ranks as in [6], we will compare the maximal column rank when the matrix is considered over different algebraic systems. For example, we can consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

as a matrix over $\mathbb{R}$, the real numbers, or as a matrix over $\mathbb{Z}_2$ , the integers modulo 2. Considered as a matrix over $\mathbb{R}$, the maximal column

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rank of **A** is three, while considered as a matrix over $\mathbb{Z}_2$, the maximal column rank of **A** is two (the third column is the sum of the first two). More generally, given semirings **S** and **T**, suppose that **A** is a matrix which can be considered either as a matrix over **S** or as a matrix over **T**. Under what circumstances is the maximal column rank of **A** over **S** equal to the maximal column rank of **A** over **T**? Is there any relationship? We will investigate these questions for several well-studied semirings, including the reals, the nonnegative reals, the integers, the nonnegative integers, the integers modulo $a$, finitely generated Boolean algebras, and fuzzy sets.

In section 2, we give necessary definitions and preliminary results, and in section 3, we establish some general inequalities for maximal column rank function, while in section 4, we obtain the cases of equality of maximal column ranks. In section 5, we compare the maximal column rank of a given $m \times n$ matrix over both a semiring and its subsemiring as $m$ and $n$ vary.

## 2. Definitions and preliminaries

A *semiring* consists of a set **S** and two binary operations on **S**, addition and multiplication, such that:

(1) **S** is an abelian monoid under addition (identity denoted by 0)

(2) **S** is a monoid under multiplication (identity denoted by 1)

(3) Multiplication distributes over addition; and

(4) s0=0s=0 for all s in **S**.

Usually **S** denotes both the semiring and the set. Many combinatorially interesting semirings are the nonnegative integers $\mathbb{Z}^+$, the nonnegative reals $\mathbb{R}^+$. The *Boolean algebra* [9] of subsets of a $k$-set, denoted $B_k$, is also a semiring, where addition corresponds to set union and multiplication corresponds to set intersection. In the sequel, we will often want to consider $B_k$ to be a subsemiring of $B_j$ when $k \leq j$. This is easily accomplished by considering the $j$-set for $B_j$ to be $\{a_1, a_2, \ldots, a_j\}$ and then associating $B_k$ with the isomorphic subsemiring of $B_j$ consisting of the set of all unions and intersections of $\{a_1\}, \{a_2\}, \ldots, \{a_{k-1}\}$ and $\{a_k, \ldots, a_j\}$. Henceforth we will assume that $B_k$ is a subsemiring of $B_j$ whenever $k \leq j$.

Let **S** be any set of two or more elements. If **S** is totally ordered by

$\leq$, that is, $\mathbf{S}$ is a *chain* under $\leq$, then define $x+y$ as $\max\{x,y\}$ and $xy$ as $\min\{x,y\}$ for all $x,y$ in $\mathbf{S}$. If $\mathbf{S}$ has a universal lower bound and a universal upper bound, then $\mathbf{S}$ becomes a semiring: a *chain semiring* [2]. In particular, the chain semiring generated by the numbers in the interval [0,1] is denoted $\mathbb{F}$, and is called the *fuzzy numbers* [8]. As above for the Boolean semirings, a chain semiring that is a subset of another may be considered a subsemiring by appending the zero and identity of the larger to the smaller. Henceforth we will assume that a chain semiring that is a subset of another is a subsemiring.

Given any semiring $\mathbf{S}$, we denote the set of $m \times n$ matrices with entries in $\mathbf{S}$ by $M_{m,n}(\mathbf{S})$. Addition of vectors ($m\times 1$ matrices), addition and multiplication of matrices, and scalar multiplication are defined as if $\mathbf{S}$ were a field. A set of vectors is a *semimodule* [3] if it is closed under addition and scalar multiplication. A subset $\mathbf{W}$ of a semimodule $\mathbf{V}$ is a *spanning set* if each vector in $\mathbf{V}$ can be written as a sum of scalar multiples (i.e., a linear combination) of elements of $\mathbf{W}$. As for real fields, we can define three concepts of rank for a matrix $\mathbf{A} \in M_{m,n}(\mathbf{S})$.

The *semiring rank* [1] of $\mathbf{A}$, $r_{\mathbf{S}}(\mathbf{A})$, is the minimum integer $k$ such that $\mathbf{A}$ can be factored as $\mathbf{A} = \mathbf{XY}$, where $\mathbf{X} \in M_{m,k}(\mathbf{S})$ and $\mathbf{Y} \in M_{k,n}(\mathbf{S})$.

The *column space* of a matrix $\mathbf{A}$ is the semimodule spanned by the columns of $\mathbf{A}$. Since the column space is spanned by a finite set of vectors, it contains a spanning set of minimum cardinality; that cardinality is the *column rank* [2] of $\mathbf{A}$, $c_{\mathbf{S}}(\mathbf{A})$.

A set $\mathbf{U}$ of vectors over $\mathbf{S}$ is *linearly dependent* if for some $u \in \mathbf{U}$, $u$ is a linear combination of another elements of $\mathbf{U}$. Otherwise $\mathbf{U}$ is *linearly independent.* The *maximal column rank* [6] of $\mathbf{A}$, $m_{\mathbf{S}}(\mathbf{A})$, is the maximal number of linearly independent columns of $\mathbf{A}$. Our goal here is to compare the values of $m_{\mathbf{S}}(\mathbf{A})$ as $\mathbf{S}$ varies over some familiar semirings such as $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}_a$ and $B_k$. We give an example:

EXAMPLE 2.1. Let $\mathbf{A} = [2 \quad 3 \quad 4 \quad 5]$ be a matrix in $M_{1,4}(\mathbb{Z}^+)$. Then we have $r_{\mathbf{Z}^+}(\mathbf{A}) = 1$. But $c_{\mathbf{Z}^+}(\mathbf{A}) = 2$ since the first two columns span the column space of $\mathbf{A}$. And $m_{\mathbf{Z}^+}(\mathbf{A}) = 3$ since the last three columns are the maximal linearly independent columns of $\mathbf{A}$.

The following result is easily established.

PROPOSITION 2.2. *Suppose that* $\mathbf{S}$ *is a semiring, and that* $\mathbf{A}$ *is a*

*$p \times q$ matrix over* $\mathbf{S}$. *If* $\mathbf{X} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}$, *where the zero block on the diagonal has arbitrary dimensions, then* $c_{\mathbf{S}}(\mathbf{A}) = c_{\mathbf{S}}(\mathbf{X})$ *and* $m_{\mathbf{S}}(\mathbf{A}) = m_{\mathbf{S}}(\mathbf{X})$.

## 3. The inequality cases in maximal column rank

In this section, we establish some general theorems about the maximal column ranks of matrices whose entries lie in two related semirings.

Suppose that $\mathbf{S}$ and $\mathbf{T}$ are semirings and $h : \mathbf{S} \longrightarrow \mathbf{T}$ is a semiring homomorphism. We identify an $m \times n$ matrix $\mathbf{A}=[a_{ij}]$ whose entries lie in $\mathbf{S}$, with the $m \times n$ matrix $H(\mathbf{A})$ whose $(i,j)$-th entry equals $h(a_{ij})$. Thus

$$H : M_{m,n}(\mathbf{S}) \longrightarrow M_{m,n}(\mathbf{T}),$$

and any matrix $\mathbf{A} \in M_{m,n}(\mathbf{S})$ can be viewed as a matrix $H(\mathbf{A}) \in M_{m,n}(\mathbf{T})$. Our first result can be summarized as follows: a homomorphism does not increase the maximal column rank.

THEOREM 3.1. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be semirings and* $h : \mathbf{S} \longrightarrow \mathbf{T}$ *be a semiring homomorphism. Then* $m_{\mathbf{S}}(\mathbf{A}) \geq m_{\mathbf{T}}(H(\mathbf{A}))$ *for every matrix* $\mathbf{A} \in M_{m,n}(\mathbf{S})$.

*Proof.* Let $m_{\mathbf{S}}(\mathbf{A}) = k$ and let $\mathbf{A} = [\mathfrak{a}_1|\mathfrak{a}_2|\dots|\mathfrak{a}_n]$, where $\mathfrak{a}_i$ is a column vector of $\mathbf{A}$ for each $i = 1, 2, \dots, n$. Put

$$H(\mathbf{A}) = [H(\mathfrak{a}_1)|H(\mathfrak{a}_2)|\dots|H(\mathfrak{a}_n)],$$

where each $H(\mathfrak{a}_i)$ represents the entrywise image vector of $\mathfrak{a}_i$ under $h$. We will show that the maximal number of linearly independent columns of $H(\mathbf{A})$ is at most $k$. To show this, let us choose any $k+1$ column vectors $H(\mathfrak{a}_{i_1}), H(\mathfrak{a}_{i_2}), \dots, H(\mathfrak{a}_{i_{k+1}})$ of $H(\mathbf{A})$. Then the column vectors $\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \dots, \mathfrak{a}_{i_{k+1}}$ of $\mathbf{A}$ are linearly dependent over $\mathbf{S}$ by the maximality of $k$. Hence there exists at least one $\mathfrak{a}_{i_p}$ among them such that

$$\mathfrak{a}_{i_p} = r_1\mathfrak{a}_{i_1} + r_2\mathfrak{a}_{i_2} + \dots + r_{p-1}\mathfrak{a}_{i_{p-1}} + r_{p+1}\mathfrak{a}_{i_{p+1}} + \dots + r_{k+1}\mathfrak{a}_{i_{k+1}}$$

with $r_i \in \mathbf{S}$.

Since $h$ is a semiring homomorphism, so is $H$. Thus

$$\begin{aligned} H(\mathfrak{a}_{i_p}) &= H(r_1\mathfrak{a}_{i_1}) + H(r_2\mathfrak{a}_{i_2}) + \ldots + H(r_{p-1}\mathfrak{a}_{i_{p-1}}) + H(r_{p+1}\mathfrak{a}_{i_{p+1}}) \\ &\quad + \ldots + H(r_{k+1}\mathfrak{a}_{i_{k+1}}) \\ &= h(r_1)H(\mathfrak{a}_{i_1}) + h(r_2)H(\mathfrak{a}_{i_2}) + \ldots + h(r_{p-1})H(\mathfrak{a}_{i_{p-1}}) \\ &\quad + h(r_{p+1})H(\mathfrak{a}_{i_{p+1}}) + \ldots + h(r_{k+1})H(\mathfrak{a}_{i_{k+1}}). \end{aligned}$$

Since each $h(r_i) \in \mathbf{T}$, we have

$$\{H(\mathfrak{a}_{i_1}), H(\mathfrak{a}_{i_2}), \ldots , H(\mathfrak{a}_{i_{k+1}})\}$$

is a linearly dependent set over $\mathbf{T}$. Thus the maximal number of linearly independent columns of $H(\mathbf{A})$ is not greater than $k$. That is, $m_{\mathbf{S}}(\mathbf{A}) \geq m_{\mathbf{T}}(H(\mathbf{A}))$. □

If $\mathbf{S}$ is a subsemiring of $\mathbf{T}$, then the canonical injection of $\mathbf{S}$ into $\mathbf{T}$ is a homomorphism, and hence by Theorem 3.1, $m_{\mathbf{S}}(\mathbf{A}) \geq m_{\mathbf{T}}(\mathbf{A})$ for each matrix $\mathbf{A} \in M_{m,n}(\mathbf{S})$. In this case, we abbreviate the above to $m_{\mathbf{S}} \geq m_{\mathbf{T}}$.

COROLLARY 3.2. *If* $\mathbf{S}$ *is a subsemiring of* $\mathbf{T}$*, then* $m_{\mathbf{S}} \geq m_{\mathbf{T}}$*. In particular,*

(1) *if* $j \geq k$*, then* $m_{B_k} \geq m_{B_j}$
(2) $m_{\mathbf{Z}^+} \geq m_{\mathbf{Z}}$, $m_{\mathbf{Z}} \geq m_P$, $m_P \geq m_{\mathbb{R}}$ *for any subring* $P$ *of the reals with identity and*
(3) $m_{\mathbf{Z}^+} \geq m_{P^+}$, $m_{P^+} \geq m_{\mathbb{R}^+}$ *for any subsemiring* $P^+$ *with identity of* $\mathbb{R}^+$.

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix whose entries belong to a semiring $\mathbf{S}$. We define the *pattern* of $\mathbf{A}$ to be the $m \times n$ matrix $\bar{\mathbf{A}} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = 0$ if $a_{ij} = 0$ and $\bar{a}_{ij} = 1$ otherwise.

COROLLARY 3.3. *Let* $\mathbf{S}$ *be an antinegative semiring (that is, only 0 has an additive inverse) with identity 1 and* $B_1$ *be the two element Boolean algebra* $\{0, 1\}$*. Then* $m_{B_1}(\bar{\mathbf{A}}) \leq m_{\mathbf{S}}(\mathbf{A})$ *for all matrices* $\mathbf{A} \in M_{m,n}(\mathbf{S})$*. In particular, if* $\mathbf{A}$ *is any (0,1) matrix, then* $m_{B_1}(\mathbf{A}) \leq m_{\mathbf{S}}(\mathbf{A})$.

*Proof.* The mapping $h : \mathbf{S} \longrightarrow B_1$ defined by $h(a) = 0$ if $a = 0$ and $h(a) = 1$ if $a \neq 0$, is a semiring homomorphism. The result now follows from Theorem 3.1. □

EXAMPLE 3.4. Let $\mathbf{S} = \mathbb{R}^+$ and let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Then $H(\mathbf{A}) = \bar{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and therefore $m_{\mathbb{R}^+}(\mathbf{A}) = 2$ since any two columns are linearly independent but the second column can be spanned by the others. Clearly, $m_{B_1}(\overline{\mathbf{A}}) = 1$. This shows that strict inequality can hold in Corollary 3.3.

COROLLARY 3.5. *Let $a$ and $b$ be integers with $a, b \geq 2$. Suppose* $\mathbf{A}$ *is a matrix with entries in $\{0, 1, 2, ..., a-1\}$. Then*

$$m_{\mathbb{Z}_a}(\mathbf{A}) \leq m_{\mathbb{Z}_{ab}}(\mathbf{A}) \leq m_{\mathbb{Z}}(\mathbf{A}).$$

*Proof.* Consider the canonical mapping $h : \mathbb{Z} \longrightarrow \mathbb{Z}_{ab}$ and $k : \mathbb{Z}_{ab} \longrightarrow \mathbb{Z}_a$. Then $h$ and $k$ are ring homomorphisms. The results follow from Theorem 3.1. □

## 4. The equality cases in maximal column rank

In section 3, we obtained some general inequalities for maximal column ranks of matrices over various semirings. We find certain cases such that the given matrix has the same maximal column rank over different semirings. In this section, we discuss the equality cases for some types of semirings.

THEOREM 4.1. *If $\mathbf{A} \in M_{m,n}(\mathbb{Z})$, then we have $m_{\mathbb{Z}}(\mathbf{A}) = m_{\mathbb{R}}(\mathbf{A})$.*

*Proof.* It is well known that the column rank of $\mathbf{A}$ over $\mathbb{Z}$ equals the column rank of $\mathbf{A}$ over $\mathbb{R}$. Since, over $\mathbb{R}$, $c(\mathbf{A}) = m(\mathbf{A})$, we will show that $m_{\mathbb{Z}}(\mathbf{A}) = c_{\mathbb{Z}}(\mathbf{A})$. In general $m_{\mathbb{Z}}(\mathbf{A}) \geq c_{\mathbb{Z}}(\mathbf{A})$. If $c_{\mathbb{Z}}(\mathbf{A}) = k$, then the column space of $\mathbf{A}$ has dimension $k$. So any $k+1$ columns of $\mathbf{A}$ are linearly dependent over $\mathbb{Z}$. Hence $m_{\mathbb{Z}}(\mathbf{A}) \leq k$. □

THEOREM 4.2. *Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be chain semirings such that $\mathbb{C}_1$ is a subsemiring of $\mathbb{C}_2$. If $\mathbf{A} \in M_{m,n}(\mathbb{C}_1)$, then $m_{\mathbb{C}_1}(\mathbf{A}) = m_{\mathbb{C}_2}(\mathbf{A})$.*

*Proof.* Since $\mathbb{C}_1 \subset \mathbb{C}_2$, we have

$$m_{\mathbb{C}_1}(\mathbf{A}) \geq m_{\mathbb{C}_2}(\mathbf{A}) \tag{1}$$

by Corollary 3.2. Let $\mathbb{C}(\mathbf{A})$ be the chain semiring consisting of 0,1 and the entries in $\mathbf{A}$. Then $\mathbb{C}(\mathbf{A})$ is a chain semiring such that $\mathbb{C}(\mathbf{A}) \subset \mathbb{C}_1$. It follows that

$$m_{\mathbb{C}(\mathbf{A})}(\mathbf{A}) \geq m_{\mathbb{C}_1}(\mathbf{A}) \tag{2}$$

by Corollary 3.2. Let $h : \mathbb{C}_2 \longrightarrow \mathbb{C}(\mathbf{A})$ be the map such that

$$h(a) = \sum_{b \in U(a)} b$$

for $a \in \mathbb{C}_2$, where

$$U(a) = \{b \in \mathbb{C}(\mathbf{A}) | b \leq a\}.$$

If $a_1, a_2 \in \mathbb{C}_2$ with $a_1 \leq a_2$, we find that $U(a_1) \subseteq U(a_2)$ and hence $h(a_1) \leq h(a_2)$. It follows that $h$ is a homomorphism from $\mathbb{C}_2$ to $\mathbb{C}(\mathbf{A})$. By the construction, $H(\mathbf{A}) = \mathbf{A}$. Therefore, we have

$$m_{\mathbb{C}_2}(\mathbf{A}) \geq m_{\mathbb{C}(\mathbf{A})}(H(\mathbf{A})) = m_{\mathbb{C}(\mathbf{A})}(\mathbf{A}) \tag{3}$$

by Theorem 3.1. Hence (1),(2) and (3) imply that $m_{\mathbb{C}_1}(\mathbf{A}) = m_{\mathbb{C}_2}(\mathbf{A})$. □

THEOREM 4.3. *Suppose that $j \leq k$, so that $B_j \subset B_k$. If $\mathbf{A} \in M_{m,n}(B_j)$, then $m_{B_j}(\mathbf{A}) = m_{B_k}(\mathbf{A})$.*

*Proof.* Since $B_j \subset B_k$ , $m_{B_j}(\mathbf{A}) \geq m_{B_k}(\mathbf{A})$. Let $B(\mathbf{A})$ be the Boolean algebra generated by 0 (empty set), 1 (the $j$-set) and the entries of $\mathbf{A}$. Then $B(\mathbf{A}) \subset B_j$. Thus $m_{B(\mathbf{A})}(\mathbf{A}) \geq m_{B_j}(\mathbf{A})$ by Corollary 3.2. By the analogous method of the proof in Theorem 4.2, we have $m_{B_j}(\mathbf{A}) = m_{B_k}(\mathbf{A})$. □

COROLLARY 4.4. *For any (0,1) matrix $\mathbf{A}$, any chain semiring $\mathbb{C}$ which contains 0,1 and any integer $k$ with $k \geq 1$, we have*

$$m_{B_1}(\mathbf{A}) = m_{B_k}(\mathbf{A}) = m_{\mathbb{C}}(\mathbf{A}) = m_{\mathbb{F}}(\mathbf{A})$$

*where $\mathbb{F}$ is the semiring of fuzzy numbers.*

*Proof.* By Theorem 4.3, $m_{B_1}(\mathbf{A}) = m_{B_k}(\mathbf{A})$ for any $k \geq 1$. Since the Boolean algebra $B_1$ is also a chain semiring and it is contained in any chain semiring $\mathbb{C}$, and in particular, in the semiring $\mathbb{F}$ of fuzzy numbers. Thus by Theorem 4.2, both $m_{\mathbb{C}}(\mathbf{A})$ and $m_{\mathbb{F}}(\mathbf{A})$ are equal to $m_{B_1}(\mathbf{A})$. □

## 5. The comparison of maximal column ranks

In this section, we compare the maximal column rank of a given matrix over both a semiring and its subsemiring.

Suppose that $\mathbf{S}$ is a subsemiring of a semiring $\mathbf{T}$. Let $F(\mathbf{S}, \mathbf{T}, m, n)$ denote the maximum integer k such that there exists a matrix in $M_{m,n}(\mathbf{S})$ with maximal column rank $k$ and for every $\mathbf{A} \in M_{m,n}(\mathbf{S})$ with $m_{\mathbf{S}}(\mathbf{A}) \leq k$ we have $m_{\mathbf{S}}(\mathbf{A}) = m_{\mathbf{T}}(\mathbf{A})$. Then $F(\mathbf{S}, \mathbf{T}, m, n) \geq 0$ since for any semiring $\mathbf{S}$, $m_{\mathbf{S}}(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is the zero matrix. In section 4, we have obtained the following equalities:
$m_{B_j} = m_{B_k}$ for any $j \leq k$, and $m_{\mathbb{C}_1} = m_{\mathbb{C}_2}$ for any chain semirings $\mathbb{C}_1$ and $\mathbb{C}_2$ with $\mathbb{C}_1 \subseteq \mathbb{C}_2$.

Also, we have shown that for any matrix $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$,

$$m_{B_1}(\bar{\mathbf{A}}) \leq m_{\mathbb{R}^+}(\mathbf{A}) \tag{4}$$

where $\bar{\mathbf{A}}$ is the pattern matrix of $\mathbf{A}$. And we have

$$m_{\mathbb{R}} \leq m_{\mathbb{R}^+} \tag{5}$$

$$m_{\mathbb{R}^+} \leq m_{\mathbb{Z}^+} \tag{6}$$

$$m_{\mathbb{Z}_a} \leq m_{\mathbb{Z}_{ab}} \tag{7}$$

for any positive integers $a$ and $b$,

$$m_{\mathbb{Z}_a} \leq m_{\mathbb{Z}} \tag{8}$$

for any positive integer $a$.

We will show that equality does not hold in general for any of (4)-(8). Our approach will be to investigate the values of $F(\mathbf{S}, \mathbf{T}, m, n)$ for appropriate semirings $\mathbf{S}$ and $\mathbf{T}$. First we will look at (4).

EXAMPLE 5.1. Let

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then $m_{B_1}(\mathbf{X}) = 2$ since the last two columns generate the first column. Thus the maximal number of linearly independent columns is 2, while $m_{\mathbb{R}^+}(\mathbf{X}) = 3$ since the three columns of $\mathbf{X}$ are linearly independent over $\mathbb{R}^+$. Thus $m_{B_1}(\mathbf{X}) < m_{\mathbb{R}^+}(\mathbf{X})$.

THEOREM 5.2. *Suppose that* $\mathbf{A} \in M_{m,n}(B_1)$.

(1) *If* $\min(m,n) \leq 2$, *then* $m_{B_1}(\mathbf{A}) = m_{\mathbb{R}^+}(\mathbf{A})$.

(2) *Let* $\min(m,n) \geq 3$. *Then* $m_{B_1}(\mathbf{A}) = m_{\mathbb{R}^+}(\mathbf{A})$ *whenever* $m_{\mathbb{R}^+}(\mathbf{A}) \leq 2$, *while there exists an* $m \times n$ *matrix* $Y \in M_{m,n}(B_1)$ *such that* $m_{B_1}(Y) < m_{\mathbb{R}^+}(Y)$ *whenever* $m_{\mathbb{R}^+}(Y) \geq 3$.

*Proof.* (1) If $m = 1$ or $n = 1$, then it is clear. Suppose that $m \geq n = 2$. Let $\mathbf{A} = [\mathfrak{a}_1|\mathfrak{a}_2]$ be an $m \times 2$ matrix. If $m_{B_1}(\mathbf{A}) = 1$, then $\mathfrak{a}_1 = b\mathfrak{a}_2$ or $\mathfrak{a}_2 = b\mathfrak{a}_1$ for $b \in B_1$. But $b = 0$ or 1, so $m_{\mathbb{R}^+}(\mathbf{A}) = 1$. If $m_{B_1}(\mathbf{A}) = 2$, then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are linearly independent over $B_1$. Thus there exists at least one positive integer $j$ such that $a_{j1} \neq a_{j2}$. This implies that $m_{\mathbb{R}^+}(\mathbf{A}) = 2$. Since we have $m_{B_1}(\mathbf{A}) \leq m_{\mathbb{R}^+}(\mathbf{A})$ by Corollary 3.3, it follows that $m_{B_1}(\mathbf{A}) = m_{\mathbb{R}^+}(\mathbf{A})$.

Suppose $n \geq m = 2$. Let $\mathbf{A} = [\mathfrak{a}_1|\mathfrak{a}_2|\dots|\mathfrak{a}_n]$ be a $2 \times n$ matrix. Since each entry of $\mathbf{A}$ is either 0 or 1, the only forms of $\mathfrak{a}_i$ are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus the maximal column rank of $\mathbf{A}$ on $\mathbb{R}^+$ is at most 2. Therefore, if $m_{B_1}(\mathbf{A}) = 2$, then $\mathbf{A}$ has at least two columns among the above three nonzero columns. It follows that $m_{\mathbb{R}^+}(\mathbf{A}) = 2$. Also if $m_{B_1}(\mathbf{A}) = 1$, then there exists one nonzero column $\mathfrak{a}_j$ such that $\mathfrak{a}_i = b_i\mathfrak{a}_j$ with $b_i \in B_1$. But $b_i = 0$ or 1, so $m_{\mathbb{R}^+}(\mathbf{A}) = 1$. Using $m_{B_1}(\mathbf{A}) \leq m_{\mathbb{R}^+}(\mathbf{A})$ by Corollary 3.3, we have $m_{B_1}(\mathbf{A}) = m_{\mathbb{R}^+}(\mathbf{A})$.

(2) If $m_{\mathbb{R}^+}(\mathbf{A}) = 1$, then clearly $m_{B_1}(\mathbf{A}) = 1$, and vice verse. If $m_{\mathbb{R}^+}(\mathbf{A}) = 2$, then $m_{B_1}(\mathbf{A}) \leq 2$ by Corollary 3.3. But $m_{B_1}(\mathbf{A}) = 1$ if and only if $m_{\mathbb{R}^+}(\mathbf{A}) = 1$. So $m_{B_1}(\mathbf{A}) = 2$.

Now, consider the matrix $\mathbf{X}$ in Example 5.1 and let

$$Y = \begin{bmatrix} \mathbf{X} & 0 \\ 0 & 0 \end{bmatrix} \oplus I_r,$$

where $I_r$ is an $r \times r$ identity matrix and the zero blocks are suitable zero submatrices such that Y is an $m \times n$ matrix. Then Y is the desired one.

For example, we can construct an $10 \times 9$ matrix Y over $B_1$ such that $m_{\mathbb{R}^+}(Y) = 6 > 5 = m_{B_1}(Y)$ as follows;

$$Y = \begin{bmatrix} \mathbf{X} & 0_1 \\ 0_2 & 0_3 \end{bmatrix} \oplus I_3,$$

with a $4 \times 3$ zero submatrix $0_3$. □

COROLLARY 5.3. *Suppose that* $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$. *If* $\min(m,n) = 1$, *then* $m_{B_1}(\bar{\mathbf{A}}) = m_{\mathbb{R}^+}(\mathbf{A})$. *On the other hand, if* $\min(m,n) \geq 2$, *then there is a matrix* $Y \in M_{m,n}(\mathbb{R}^+)$ *such that* $m_{B_1}(\bar{Y}) < m_{\mathbb{R}^+}(Y)$, *whenever* $m_{\mathbb{R}^+}(Y) \geq 2$.

*Proof.* For the case $\min(m,n) = 1$, it is clear. Consider a matrix $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ over $\mathbb{R}^+$. Then $m_{\mathbb{R}^+}(X) = 2$ but $m_{B_1}(\bar{X}) = 1$. Thus we can construct an $m \times n$ matrix Y such that $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \oplus I_r$ as the case of Theorem 5.2 (2). □

The following example gives some insight into inequality (5.3).

EXAMPLE 5.4. Let $\mathbf{A} = [3 \quad 5]$ be a $1 \times 2$ matrix. Then the second column of $\mathbf{A}$ is 5/3 times the first, so there is only one linearly independent column in $\mathbf{A}$ over $\mathbb{R}^+$ and hence $m_{\mathbb{R}^+}(\mathbf{A}) = 1$.

However, no integer multiple of the first column equals the second, and vice versa, so the two columns of $\mathbf{A}$ are linearly independent over $\mathbb{Z}^+$ and hence we have that $m_{\mathbb{Z}^+}(\mathbf{A}) = 2 > m_{\mathbb{R}^+}(\mathbf{A})$.

Thus we have the following comparison theorem.

THEOREM 5.5. $F(\mathbb{Z}^+, \mathbb{R}^+, m, n) = 1$.

*Proof.* Since we know that any nonzero matrix has maximal column rank at least 1, it is clear that $m_{\mathbb{R}^+}(\mathbf{A}) = m_{\mathbb{Z}^+}(\mathbf{A})$ whenever $m_{\mathbb{Z}^+}(\mathbf{A}) \leq 1$. But we had a matrix $\mathbf{A}$ in Example 5.4 such that $m_{\mathbb{R}^+}(\mathbf{A}) = 1$ and $m_{\mathbb{Z}^+}(\mathbf{A}) = 2$. By proposition 2.2, we always have an $m \times n$ matrix $\mathbf{X}$

over $\mathbb{Z}^+$ with $m_{\mathbb{R}^+}(\mathbf{X}) = 1$ and $m_{\mathbb{Z}^+}(\mathbf{X}) = 2$ whenever $n \geq 2$. Thus $F(\mathbb{Z}^+, \mathbb{R}^+, m, n) = 1$. □

Let $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$. If $m_{\mathbb{R}}(\mathbf{A}) = 1$, then each column of $\mathbf{A}$ is a multiple of the first nonzero column of $\mathbf{A}$. Moreover, each column of $\mathbf{A}$ is a nonnegative multiple of that column, and hence there exist no two linearly independent columns of $\mathbf{A}$. Consequently $m_{\mathbb{R}^+}(\mathbf{A}) = 1$ and hence we have $F(\mathbb{R}^+, \mathbb{R}, m, n) \geq 1$. We establish further result.

THEOREM 5.6. *Let* $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ *with* $\min(m, n) \geq 2$. *If* $m_{\mathbb{R}}(\mathbf{A}) = 2$, *then* $m_{\mathbb{R}^+}(\mathbf{A}) = 2$.

*Proof.* Suppose that $m_{\mathbb{R}}(\mathbf{A}) = 2$. Then the number of maximal linearly independent columns of $\mathbf{A}$ is 2 over $\mathbb{R}$. That is, for any three columns $\mathfrak{a}_i, \mathfrak{a}_j$ and $\mathfrak{a}_k$ of $\mathbf{A}$, we have $x\mathfrak{a}_i + y\mathfrak{a}_j + z\mathfrak{a}_k = 0$ for some $x, y, z \in \mathbb{R}$. Since $\mathfrak{a}_i, \mathfrak{a}_j$ and $\mathfrak{a}_k$ have nonnegative entries, one of $x, y$ and $z$ is positive while another is negative. Without loss of generality, we may assume that $x$ is positive and the others are negative. Then $\mathfrak{a}_i = (-y/x)\mathfrak{a}_j + (-z/x)\mathfrak{a}_k$. Since $(-y/x)$ and $(-z/x)$ are positive, $\mathfrak{a}_i, \mathfrak{a}_j$ and $\mathfrak{a}_k$ are linearly dependent over $\mathbb{R}^+$. Thus three arbitrary columns of $\mathbf{A}$ are linearly dependent over $\mathbb{R}^+$ and hence $m_{\mathbb{R}^+}(\mathbf{A}) \leq 2$. Using (5), the result follows. □

Now we see that there exists a matrix $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ such that $m_{\mathbb{R}}(\mathbf{A}) < m_{\mathbb{R}^+}(\mathbf{A})$ for $m_{\mathbb{R}}(\mathbf{A}) \geq 3$.

EXAMPLE 5.7. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Then $m_{\mathbb{R}}(\mathbf{A}) = 3$ since any three columns of $\mathbf{A}$ are linearly independent over $\mathbb{R}$ but $\mathfrak{a}_1 + \mathfrak{a}_3 - \mathfrak{a}_2 = \mathfrak{a}_4$, where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ and $\mathfrak{a}_4$ are the columns of $\mathbf{A}$ by turns. From Corollary 3.3, we have $4 = m_{B_1}(\mathbf{A}) \leq m_{\mathbb{R}^+}(\mathbf{A})$. Hence $m_{\mathbb{R}}(\mathbf{A}) < m_{\mathbb{R}^+}(\mathbf{A})$.

Using Proposition 2.2, we can obtain a matrix $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ such that $m_{\mathbb{R}}(\mathbf{A}) < m_{\mathbb{R}^+}(\mathbf{A})$ for $\min(m, n) \geq 4$.

LEMMA 5.8. *Let* $\mathbf{S}$ *be a field. If* $\mathbf{A} \in M_{m,n}(\mathbf{S})$ *and* $\min(m,n) = k$, *then* $m_{\mathbf{S}}(\mathbf{A}) \leq k$.

*Proof.* In a field, the maximal column rank and the field rank are the same. So $m_{\mathbf{S}}(\mathbf{A}) \leq k$. □

Using Theorem 5.6, Example 5.7 and Proposition 2.2, we have the following comparison theorem.

THEOREM 5.9.

$$F(\mathbb{R}^+, \mathbb{R}, m, n) = \begin{cases} 1 & \text{if} \quad \min(m,n) = 1 \\ 2 & \text{if} \quad \min(m,n) = 2 \\ 3 & \text{otherwise} \end{cases}$$

*Proof.* If $\min(m,n) = 1$, then it is trivial. Suppose that $\min(m,n) = 2$. For any $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ with $m_{\mathbf{R}^+}(\mathbf{A}) \leq 1$, it is clear that $m_{\mathbf{R}}(\mathbf{A}) = m_{\mathbf{R}^+}(\mathbf{A})$. And we have $m_{\mathbf{R}}(\mathbf{A}) \leq 2$ by Lemma 5.8. Assume $m_{\mathbf{R}^+}(\mathbf{A}) = 2$. Then $m_{\mathbf{R}}(\mathbf{A}) \leq 2$. But $m_{\mathbf{R}}(\mathbf{A})$ can be neither 0 nor 1. Thus $m_{\mathbb{R}}(\mathbf{A}) = 2$. Then $m_{\mathbb{R}^+}(\mathbf{A}) = m_{\mathbf{R}}(\mathbf{A})$ by Theorem 5.6. So this case holds.

Now, let $\min(m,n) \geq 3$. By Theorem 5.6, it suffices to show that for any $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ with $m_{\mathbb{R}^+}(\mathbf{A}) = 3$, $m_{\mathbb{R}^+}(\mathbf{A}) = m_{\mathbf{R}}(\mathbf{A})$. Assume that $m_{\mathbf{R}^+}(\mathbf{A}) = 3$. Then $m_{\mathbb{R}}(\mathbf{A}) \leq 3$ by (5). But $m_{\mathbf{R}}(\mathbf{A})$ cannot be 0, 1 and 2 by Theorem 5.6. Thus $m_{\mathbb{R}}(\mathbf{A}) = 3$. Thus for any $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ with $m_{\mathbf{R}^+}(\mathbf{A}) \leq 3$, we have $m_{\mathbb{R}}(\mathbf{A}) = m_{\mathbf{R}^+}(\mathbf{A})$. Further, Example 5.7 and Proposition 2.2 imply that there exists a matrix $\mathbf{A} \in M_{m,n}(\mathbb{R}^+)$ such that $m_{\mathbb{R}^+}(\mathbf{A}) > m_{\mathbb{R}}(\mathbf{A})$. Hence we have

$$F(\mathbb{R}^+, \mathbb{R}, m, n) = 3.$$ □

Next we show that equality need not hold in (7) and (8).

EXAMPLE 5.10. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ a-1 & a-2 \end{bmatrix} \in M_{2,2}(\mathbb{Z}_a)$$

with $a \geq 3$. Then $m_{\mathbf{Z}_a}(\mathbf{A}) = 1$ but $m_{\mathbf{Z}_{ab}}(\mathbf{A}) = 2$ for any $b \geq 2$ and $m_{\mathbf{Z}}(\mathbf{A}) = 2$.

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