

ON A CLASS OF OPERATORS

by YOUNGOH YANG

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Abstract. In this paper we show that the Weyl spectrum of an operator of class W satisfies the spectral mapping theorem for analytic functions and give the equivalent conditions for an operator of the form normal + compact to be polynomially compact.

0. Introduction. Let H be an infinite dimensional Hilbert space and let $B(H)$ be the set of all bounded linear operators on H . If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T . An operator $T \in B(H)$ is said to be *Fredholm* if its range $\text{ran } T$ is closed and both the null space $\ker T$ and $\ker T^*$ are finite dimensional. The *index* of a Fredholm operator T , denoted by $i(T)$, is defined by

$$i(T) = \dim \ker T - \dim \ker T^*.$$

It is well known ([3]) that $i: \mathcal{F} \rightarrow \mathbb{Z}$ is a continuous function, where the set \mathcal{F} of Fredholm operators has the norm topology and \mathbb{Z} has the discrete topology. The *essential spectrum* of T , denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl operator*. The *Weyl spectrum* of T , denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([1]) that for any operator T , $\sigma_e(T) \subset \omega(T) \subset \sigma(T)$ and $\omega(T)$ is a nonempty compact subset of \mathbb{C} .

We say that an operator $T \in B(H)$ is of class W if $\sigma_e(T) = \omega(T)$. For example, every normal, compact, quasinilpotent operator is of class W . However, consider the unilateral shift U on l_2 given by

$$U(x_1, x_2, \dots) = (0, x_1, x_2, x_3, \dots).$$

Then U is hyponormal, $\omega(U) = \sigma(U) = D$ (= the closed unit disc) and $\sigma_e(U) = C$ (= the unit circle). Hence U is not of class W and so we note that T is not of class W , even if T is hyponormal. By [2, Theorem 4.1], every Toeplitz operator is not of class W .

It is also known that the mapping $T \rightarrow \omega(T)$ is upper semi-continuous, but not continuous at T ([8]). However if $T_n \rightarrow T$ with $T_n T = T T_n$, for all $n \in \mathbb{N}$, then

$$\lim \omega(T_n) = \omega(T). \tag{1}$$

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It is known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) \subset f(\omega(T)), \quad (2)$$

by [1, Theorem 3.2].

The inclusion (2), may be proper. (See [1, Example 3.3].) If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions.

In this paper we show that the Weyl spectrum of an operator of class W satisfies the spectral mapping theorem for analytic functions and give the equivalent conditions for an operator of the form normal + compact to be polynomially compact.

1. Spectral mapping theorem. The Weyl spectrum of an operator is the disjoint union of the essential spectrum and a particular open set.

LEMMA 1. ([1],[3]) For any operator T in $B(H)$,

$$\omega(T) = \sigma_e(T) \cup \theta(T),$$

where $\theta(T) = \{\lambda : T - \lambda \text{ is Fredholm and } i(T - \lambda) \neq 0\}$. The union is disjoint.

For example, if U is the unilateral shift, then $\sigma_e(U) = \{\lambda : |\lambda| = 1\}$ and $\theta(U) = \{\lambda : |\lambda| < 1\}$. From Lemma 1, we note that every normal operator is of class W , and that $\sigma_e(T) = \omega(T)$ if and only if the open set $\theta(T)$ is empty. By [1, Example 2.12], every compact operator K is of class W . Also it is easy to show that if T is of class W and $\alpha \in C$, then T^* and αT are of class W .

For an example of a nonnormal operator of class W , consider $T = U \oplus U^*$ where U is the unilateral shift. In this case, $\omega(T) = \{\lambda : |\lambda| = 1\} = \sigma_e(T)$. Thus T is of class W . However T is not a normal operator. Thus our class is strictly larger than the class of normal operators.

It is easy to show that the set of operators of class W is closed in $B(H)$, invariant under compact perturbations and closed under similarity.

THEOREM 2. If T , in $B(H)$, is of the form normal + compact, then T is of class W .

Proof. Let $T = N + K$, where N is normal and K is compact. If T is not of class W , then by Lemma 1 there exists $\lambda \in C$ such that $T - \lambda$ is Fredholm of nonzero index. But, by [3], $T - \lambda - K$ is Fredholm and $i(T - \lambda) = i(T - \lambda - K) = i(N - \lambda) = 0$. This is a contradiction.

We note that if T is a normal operator and f is any continuous complex-valued function on $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$, and so $f(T)$ is of class W ([1]). We obtain the following similar result.

THEOREM 3. If T is of class W and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.

Proof. Suppose that p is any polynomial. Then, by the spectral mapping theorem,

$$p(\omega(T)) = p(\sigma_e(T)) = \sigma_e(p(T)) \subseteq \omega(p(T)).$$

But for any operator $T \in B(H)$, $\omega(p(T)) \subseteq p(\omega(T))$ by (2). Therefore $\omega(p(T)) = p(\omega(T))$, for any polynomial p .

If f is analytic on a neighborhood of $\sigma(T)$ then, by Runge's theorem ([3]), there is a sequence (p_n) of polynomials such that $f_n \rightarrow f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [8], we have

$$\omega(f(T)) = \lim \omega(p_n(T)) = \lim p_n(\omega(T)) = f(\omega(T)).$$

COROLLARY 4. *If T is of class W and f is analytic on a neighborhood of $\sigma(T)$, then $f(T)$ is of class W .*

Proof. By Theorem 3 and the spectral mapping theorem,

$$\omega(f(T)) = f(\omega(T)) = f(\sigma_e(T)) = \sigma_e(f(T)).$$

Thus $f(T)$ is of class W .

An operator T is said to be *polynomially compact* if there exists a polynomial p such that $p(T)$ is compact.

THEOREM 5. *For an operator T of the form normal + compact, the following are equivalent:*

- (1) T is polynomially compact;
- (2) there exists an analytic function f on $\sigma(T)$ such that $f(T)$ is compact and f has finitely many zeros on $\omega(T)$;
- (3) $\omega(T)$ is finite.

Proof. (1) \Rightarrow (2) trivially.

(2) \Rightarrow (3). By Theorems 2, 3 and [1, Example 2.12], we have $f(\omega(T)) = \omega(f(T)) = \{0\}$. Since f has only finite many zeros on $\omega(T)$, it follows that $\omega(T)$ is finite.

(3) \Rightarrow (1). Suppose that $\omega(T)$ is finite. Let p be any nonzero polynomial such that p is 0 on $\omega(T)$. By Theorems 2 and 3, we have $\omega(p(T)) = p(\omega(T)) = \{0\}$. Since T is of the form normal S + compact, $p(T)$ is an operator of the form normal + compact, say $p(T) = p(S) + K$, where K is compact and $p(S)$ is normal. Thus $\omega(p(S)) = \omega(p(T)) = p(\omega(T)) = \{0\}$ by [1, Corollary 2.7]. Since $p(S)$ is normal, it follows that $p(S)$ is compact. (See the remarks following [1, Corollary 6.3].) Thus $p(T)$ is compact and hence T is polynomially compact.

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REFERENCES

1. S. K. Berberian, The Weyl's spectrum of an operator, *Indiana Univ. Math. J.* **20**(6) (1970), 529–544.
2. L. A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13** (1966), 285–288.

3. J. B. Conway, *Subnormal operators* (Pitman, Boston, 1981).
4. B. Gramsch and D. Lay, Spectral mapping theorems for essential spectra, *Math. Ann* **192** (1971), 17–32.
5. P. R. Halmos, *A Hilbert space problem book*. (Springer-Verlag, 1984).
6. R. E. Harte, *Invertibility and singularity for bounded linear operators* (Marcel Dekker, New York, 1988).
7. T. Kato, *Perturbation theory for linear operators* (Springer-Verlag, 1966).
8. K. K. Oberai, On the Weyl spectrum, *Illinois J. Math* **18** (1974), 208–212.
9. K. K. Oberai, On the Weyl spectrum II, *Illinois J. Math.* **21** (1977), 84–90.

DEPARTMENT OF MATHEMATICS
CHEJU NATIONAL UNIVERSITY
CHEJU 690-756
KOREA
E-mail: yangyo@cheju.cheju.ac.kr

Present address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA
TUSCALOOSA
ALABAMA 35487-0350
USA
E-mail: yyang@gp.as.ua.edu