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SEMIRING RANKS AND THEIR PRESERVERS*

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ABSTRACT. We compare Boolean rank and maximal column rank on the matrices over binary Boolean algebra, nonnegative integers, nonnegative reals, and general Boolean algebra, respectively. We also characterize the linear operators that preserve maximal column rank over general Boolean matrices.

1. Introduction

There is much literature on the study of matrices over a finite Boolean algebra. But many results in Boolean matrix theory are stated only for binary Boolean matrices. This is due in part to a semiring isomorphism between the matrices over the Boolean algebra of subsets of a k element set and the k tuples of binary Boolean matrices. This isomorphism allows many questions concerning matrices over an arbitrary finite Boolean algebra to be answered using the binary Boolean case. However there are interesting results about the general Boolean matrices that have not been mentioned and they differ somewhat from the binary case.

In this paper, first we will show the extent of the difference between semiring rank and maximal column rank of matrices over a general Boolean algebra. Second, there are some unproved ones on semiring rank, column rank, and maximal column rank through the previous researches, and so we will give the solutions of them. Finally, we also obtain the characterizations of the linear operators that preserve maximal column ranks of general Boolean matrices.

2. Definitions and Preliminaries

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Let **B** be the *Boolean algebra* of the subsets of a k element set S_k and $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the singleton subsets of S_k. We write $+$ for union and denote intersection by juxtaposition;0 denotes the null set and 1 the set S_k. Under these two operations, **B** is a commutative, antinegative semiring(that is, only 0 has an additive inverse); all of its elements, except 0 and 1, are zero-divisors. Let $\mathbf{M}_{m,n}(\mathbf{B})$ denote the set of all $m \times n$ matrices with entries in **B**. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well.

Definition ([5]). *Let A be an $m \times n$ matrix over* **B**. *The p-th constituent of A, A_p is*

$$(a_p)_{st} = \begin{cases} 1 & \text{if } \sigma_p \subseteq a_{st}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Comparisons of various ranks over Boolean matrices

3.1 Boolean rank versus Boolean maximal column rank

Definition 3.1 ([2]). *Let $A(\neq 0) \in \mathbf{M}_{m,n}(\mathbf{B})$. The Boolean rank, $b(A)$ is the least index r such that $A = B_{m\times r} \cdot C_{r\times n}$.*

Let **V** be nonempty subset of $\mathbf{M}_{r,1}(\mathbf{B})$ such that it is closed under $+$ and $\cdot$ by scalars. Then **V** is called a *vector space* over **B**.

We define "subspace" and "generating sets" as the things to coincide with familiar definitions when **B** is a field. We think of $< F >$ as the subspace generated by the subset F of **V**.

As with fields, a *basis* for a vector space **V** is a generating subset of the least cardinality. That cardinality is the *dimension*, $dim(\mathbf{V})$ of **V**.

Definition 3.2 ([2]). *The Boolean column rank, $c(A)$ of $A \in \mathbf{M}_{m\times n}(\mathbf{B})$ is the dimension of the space $< A >$ generated by the columns of A.*

Definition 3.3 ([7]). *A set G of vectors over* **B** *is linearly dependent if for some $g \in G, g \in < G\backslash\{g\} >$. Otherwise, G is linearly independent.*

Definition 3.4 ([7]). *The maximal column rank , $mc(A)$ of an $m \times n$ matrix A over* **B** *is the maximal number of the columns of A which are linearly independent over* **B**.

(3.1) For all $m \times n$ matrices A over **B**,

$$0 \leq b(A) \leq mc(A) \leq n.([3])$$

(3.2) For any $p \times q$ matrix A over $\mathbb{B}$, the Boolean rank of $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is $b(A)$ and its Boolean maximal column rank is $mc(A)$.

(3.3) The Boolean rank of a matrix is the maximum of the binary Boolean ranks of its constituents. ([5])

Lemma 3.1. *For any binary Boolean matrix A, we have $mc(A) = mc_1(A)$.*

Proof. Assume $mc(A) = k$. Then $\exists$ k columns $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k$ which are linearly independent over $\mathbf{B}_1$. Consider $(\mathbf{X}_1)_p, (\mathbf{X}_2)_p, \cdots, (\mathbf{X}_k)_p$. If $(\mathbf{X}_i)_p = \sum_{j \neq i} (\mathbf{X}_j)_p$, then

$$\mathbf{X}_i = (\mathbf{X}_i)_p = \sum_{j \neq i} (\mathbf{X}_j)_p = \sum_{j \neq i} \mathbf{X}_j.$$

This contradicts to the assumption.

Thus $mc_1(A) \geq k$.

Conversely, suppose $mc_1(A) = k$. Then $\exists$ k columns, $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_k$ which are linearly independent over $\mathbf{B}_1$. If $\mathbf{Y}_i = \sum_{j \neq i} (\alpha_j) \mathbf{Y}_j$ where $\alpha_j \in \mathbf{B}_1$, then

$$\begin{aligned} \mathbf{Y}_i = (\mathbf{Y}_i)_p &= \sum_{j \neq i} (\alpha_i)_p (\mathbf{Y}_j)_p \\ &= \sum_{j \neq i} (\alpha_i)_p \mathbf{Y}_j. \end{aligned}$$

This is a contradiction. Therefore

$$mc(A) \geq k. \qquad \square$$

Definition 3.5 $[\beta(\mathbb{B}, m, n)]$. *$\beta(\mathbb{B}, m, n)$ is the largest integer r such that for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, $b(A) = mc(A)$ if $b(A) \leq r$.*

In general,

$$0 \leq \beta(\mathbb{B}, m, n) \leq n.$$

(3.4) Over any Boolean algebra $\mathbb{B}$, if $mc(A) > b(A)$ for some $p \times q$ matrix A, then for all $m \geq p$ and $n \geq q$, $\beta(\mathbb{B}, m, n) < b(A)$.

Proof. Since $mc(A) > b(A)$ for some $p \times q$ matrix A, we have $\beta(\mathbb{B}, p, q) < b(A)$ from the definition. Let $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ be an $m \times n$ matrix containing A as a submatrix. Then

$$b(B) = b(A) < mc(A) = mc(B).$$

So, $b(B) < mc(B)$. Hence

$$\beta(\mathbf{B}, m, n) < b(B),$$

for all $m \geq p$ and $n \geq q$. □

Lemma 3.2. *In* $\mathbf{B}_1$, $b_1(A) = 1$ *if and only if* $mc_1(A) = 1$.

Proof. $(\Longleftarrow)$; It is obvious.

$(\Longrightarrow)$; Suppose $b_1(A) = 1$. Then A can be split into two matrices, that is ,

$$A = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_m \end{pmatrix}_{m\times 1} \begin{pmatrix} n_1 & n_2 & \cdots & n_n \end{pmatrix}_{1\times n}$$
$$= \begin{pmatrix} m_1n_1 & \cdots & m_1n_i & \cdots & m_1n_n \\ m_2n_1 & \cdots & m_2n_i & \cdots & m_2n_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ m_mn_1 & \cdots & m_mn_i & \cdots & m_mn_n \end{pmatrix}.$$

If $\exists\ n_i, n_j \neq 0\ (i \neq j)$, then $n_i = n_j = 1$ ($\because n_i, n_j \in \mathbf{B}_1 = \{0,1\}$) So ith and jth columns of A are linearly dependent. Thus we get $mc_1(A) = 1$. □

Generally speaking, it is false that $b(A) = 1$ if and only if $mc(A) = 1$. Now, we suggest a counter-example.

Counter-example 3.3. Let $A = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ where $\sigma_1, \sigma_2, \sigma_3$ are mutually distinct.

Then since $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$,

$$b(A) = 1.$$

But it is easily obtained that $mc(A) = 3$. □

Theorem 3.1.

$$\beta(\mathbf{B}_1, m, n) = \begin{cases} 1 & \text{if } \min\{m,n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. We can have

$$\beta(\mathbf{B}_1, m, n) = 1,$$

whenever $\min\{m,n\}=1$. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. A simple computation gives

$$mc_1(A) = 4 \quad \text{and} \quad b_1(A) = 3.$$

By (3.4),

$$\beta(\mathbf{B}_1, m, n) \leq 2,$$

for all $m \geq 3$ and $n \geq 4$.

Suppose $m \geq 2$ and $n \geq 2$. Then

$$b_1(A) = 2 \quad \text{iff} \quad mc_1(A) = 2 - - - - - - - - -(*).$$

For if $mc_1(A)$=2 ,then $b_1(A)$= 1 or 2.

So we can have $b_1(A)$=2 from the above.

Conversely, suppose $b_1(A)$=2. Then $\exists\ F_{m,2}, G_{2,n}$ such that $A = FG$. For some permutation P,

$$GP = \begin{pmatrix} 1 & 0 & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad or$$

$$GP = \begin{pmatrix} 1 & 1 & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad or$$

$$GP = \begin{pmatrix} 1 & 0 & x_3 & x_4 & \cdots & x_n \\ 1 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad with \quad x_i, y_i \in \mathbf{B}_1.$$

If not, then $b_1(G)$=1 and hence $b_1(A)$=1. This is a contradiction. Hence certain two columns of F are maximal linearly independent columns of A. That is to say, $mc_1(A)$=2. Therefore we get

$$\beta(\mathbf{B}_1, m, n) \geq 2,$$

for all $\min\{m,n\} \geq 2$.

Finally we only show the case of $m \geq 3$ and n=3. Note that

$$b_1(A) = 3 \quad \text{iff} \quad mc_1(A) = 3,$$

whenever $A \in \mathbf{M}_{m,3}(\mathbf{B}_1) (m \geq 3)$.

For if $mc_1(A)$=3 ,then $b_1(A)$= 1 or 2 or 3.

But $b_1(A) \neq 1$ and 2 by $(*)$. Therefore

$$b_1(A) = 3.$$

Conversely , if $b_1(A)$=3 ,then $mc_1(A) \geq 3$ but $mc_1(A) \leq 3$. Therefore

$$mc_1(A) = 3.$$

Hence

$$\beta(\mathbf{B}_1, m, n) = \begin{cases} 1 & \text{if } \min\{m, n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

□

Theorem 3.2. *For a nonbinary Boolean algebra* $\mathbf{B}$,

$$\beta(\mathbf{B}, m, n) = \begin{cases} 0 & \text{if } n \geq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $A = (\sigma_1 \;\; \sigma_2)_{1\times 2}$, where σ_1 and σ_2 are distinct. Then $b(A) = 1$,but $mc(A) = 2$. So by (3.4), we have

$$\beta(\mathbf{B}, m, n) = 0,$$

for all $n \geq 2$.

Consider the case, n=1. Then for any $A \in \mathbb{M}_{m,1}$, $b(A) = 1$ and $mc(A) = 1$. Therefore

$$\beta(\mathbf{B}, m, 1) = 1.$$

Hence we obtain the desired result. □

3.2 Comparisons of rank, column rank and maximal column rank over Boolean matrices

In this section, we shall now discuss some proofs which are related with μ,α and β in $\mathbf{B}_1$,$\mathbb{Z}^+$,$\mathbb{F}^+$, and $\mathbb{B}$.

Theorem 3.3. *Let $\mathbf{Z}^+$ be a semiring of nonnegative integers. Then for $m \geq 1$,*

$$\beta(\mathbf{Z}^+, m, n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Proof. It is clear when n=1. Consider $A = (2\ 3)_{1\times 2}$ Then $b(A) = 1$ but $mc(A) = 2$. By (3.4),

$$\beta(\mathbf{Z}^+, m, n) = 0,$$

for all $n \geq 2$. Hence the proof is completed. □

Corollary.

$$\mu(\mathbf{Z}^+, m, n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Proof. Similarly, consider $A = (2\ 3)_{1\times 2}$. Then we can obtain the desired result from the fact that $b(A) = 1$ and $c(A) = 2$. □

Theorem 3.4. *Let $\mathbf{F}$ be a subfield of the reals, and $\mathbf{F}^+$ be the subset of $\mathbf{F}$ consisting of the nonnegative members of $\mathbf{F}$. Then*

$$\beta(\mathbf{F}^+, m, n) = \begin{cases} 1 & \text{if } \min\{m, n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. In $\mathbf{F}^+$,

$$r(A) = 1 \quad \text{iff} \quad mc(A) = 1 \quad - - - - - - - - (*).$$

The sufficient condition is obvious and so we only show that the necessary condition. Suppose $r(A) = 1$. Then there exist $F_{m\times 1}, G_{1\times n}$ such that $A = FG$. Put $F = (x_1, x_2, \cdots, x_m)^T$ and $G = (y_1, y_2, \cdots, y_n)$, where $x_i, y_i \in \mathbf{F}^+$. Then

$$\begin{aligned} A_{m\times n} &= F_{m\times 1}G_{1\times n} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} (y_1 \quad y_2 \quad \cdots \quad y_n) \\ &= \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{pmatrix}. \end{aligned}$$

Since $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} y_i = (y_i/y_j) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \cdot y_j$, $mc(A) = 1$. Therefore

$$\beta(\mathbf{F}^+, m, n) \geq 1.$$

So if $\min\{m, n\} = 1$, then it is clear that $\beta(\mathbf{F}^+, m, n) = 1$.

In $\mathbf{F}^+$,

$$r(A) = 2 \quad \text{iff} \quad mc(A) = 2.$$

For if $mc(A) = 2$, then it is trivial that $r(A) = 2$ by $(*)$. Conversely, suppose $r(A) = 2$. Then $mc(A) \geq 2$. If $mc(A) > 2$, then there exist linearly independent columns, say $\mathbf{a}_i$,$\mathbf{a}_j$ and $\mathbf{a}_k$ of A over $\mathbf{F}^+$. Since the rank of A over the subfield $\mathbf{F}$ of the reals 2, there exist scalars α,β and γ, not all zero, such that $\alpha\mathbf{a}_i + \beta\mathbf{a}_j + \gamma\mathbf{a}_k = 0$. Since all the entries in A are nonnegative, at least one of α,β and γ is positive and one negative. We may assume that two are positive(or at least nonnegative) and one negative, say γ is negative. Then $(\alpha/-\gamma)\mathbf{a}_i + (\beta/-\gamma)\mathbf{a}_j = \mathbf{a}_k$. Thus $\mathbf{a}_i$,$\mathbf{a}_j$ and $\mathbf{a}_k$ are linear dependent over $\mathbf{F}^+$ which leads to a contradiction of the fact that they are linearly independent. Hence $mc(A) = 2$. Therefore we have

$$\beta(\mathbf{F}^+, m, n) \geq 2 \quad - - - - - - - - -(**)$$

for $\min\{m, n\} \geq 2$.

If $A \in \mathbf{M}_{2,n}$ for $n \geq 2$, then $mc(A) = r(A) \leq 2$. Thus $(**)$ implies that

$$\beta(\mathbf{F}^+, 2, n) = 2 \text{ for } n \geq 2.$$

If $m \geq 3, n = 3$ and $A \in \mathbf{M}_{m,3}(F^+)$ with $mc(A) = 3$, then by $(*)$ and $(**)$, $r(A) = 3$. Therefore we obtain

$$\beta(\mathbf{F}^+, m, 3) = 3 \text{ for } m \geq 3.$$

Finally, if $m \geq 3$ and $n \geq 4$, then

$$\beta(\mathbf{F}^+, m, n) \leq 2.$$

For let $A = \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ a_1 & 0 & 0 & a_5 \\ 0 & 0 & a_4 & a_6 \end{pmatrix}$ where $a_1, a_2, \cdots, a_6 \in \mathbf{F}^+$, then $mc(A) = 4$ and $b(A) \leq 3$. Therefore from (3.4) we get,

$$\beta(\mathbf{F}^+, m, n) \leq 2.$$

Hence from (**) we have

$$\beta(\mathbb{F}^+, m, n) = 2,$$

for $m \geq 3$ and $n \geq 4$. Hence the proof is completed. □

4. Linear operators that preserve maximal column rank of the nonbinary Boolean matrices

In this section, we obtain the characterizations of the linear operators that preserve Boolean maximal column rank of the nonbinary Boolean matrices.

If $\mathbf{V}$ is a vector space over a Boolean algebra $\mathbf{B}$, a mapping $T : \mathbf{V} \to \mathbf{V}$ which preserves sums and 0 is said to be a *(Boolean) linear operator.*

Definition 4.1. *A linear operator T on $\mathbf{M}_{m,n}(\mathbf{B})$ is said to* **preserve Boolean maximal column rank** *if $mc(T(A)) = mc(A)$ for all $A \in \mathbf{M}_{m,n}(\mathbf{B})$. In particular, T* **preserves Boolean maximal column rank r** *if $mc(T(A)) = r$ whenever $mc(A) = r$.*

Similarly we can define the terms, such as Boolean rank preserver and Boolean rank r preserver.

Definition 4.2 ([9]). *Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbf{B})$. For each $1 \leq p \leq k$, a map T_p* **is called its p-th constituent** *if $T_p(B) = (T(B))_p$ for every $B \in \mathbf{M}_{m,n}(\mathbf{B}_1)$.*

Lemma 4.1 ([9]). *If $A \in \mathbf{M}_{m,n}(\mathbf{B})$ and U,V are invertible matrices, then*

$$mc(A) = mc(UA) = mc(AV).$$

Lemma 4.2. *Assume T is a linear operator on $\mathbf{M}_{m,n}(\mathbf{B})$. If T preserves Boolean maximal column rank r, then each constituent T_p preserves Boolean maximal column rank r on $\mathbf{M}_{m,n}(\mathbf{B}_1)$.*

Proof. Suppose that $A \in \mathbf{M}_{m,n}(\mathbf{B}_1)$ with $mc_1(A) = r$. By Lemma 3.3, we have

$$mc(A)(= mc_1(A)) = r, \text{ and } mc(\sigma_p A) = r,$$

for each p=1,2,···,k. since T preserves Boolean maximal column rank r, $mc(T(\sigma_p A)) = r$. But

$$\begin{aligned} r = mc(T(\sigma_p A)) &= mc(\sigma_p T(A)) \\ &= mc(\sigma_p \sum_i \sigma_i T_i(A_i)) \\ &= mc(\sigma_p T_p(A_p)) \\ &= mc(\sigma_p T_p(A)). \end{aligned}$$

Therefore $mc(\sigma_p T_p(A)) = r$ for each p=1,2,$\cdots$,k, and hence $mc_1(T_p(A)) = r$.
□

Lemma 4.3 (**[9]**). *Suppose T is a linear operator on $\mathbf{M}_{m,n}(\mathbf{B})$. If each constituent T_p preserves binary Boolean rank r, then T preserves Boolean rank r.*

Suppose T is a linear operator on $\mathbf{M}_{m,n}(\mathbf{B})$. Say that T is a

(i) *Congruence operator* if there exist invertible matrices $m \times m$ and $n \times n$ Boolean matrices U, V such that $T(A) = UAV$ for any A in $\mathbf{M}_{m,n}(\mathbf{B})$.

Let σ^* denote the complement of σ for each σ in $\mathbf{B}$.

(ii) the p-th *rotation operator*, $R^{(p)}$, on $\mathbf{M}_{m,n}(\mathbf{B})$ if

$$R^{(p)}(A) = \sigma_p A_p^t + \sigma_p^* A, \text{ for } 1 \leq p \leq k,$$

where A_p^t is the transpose matrix of A_p.

We see that $R^{(p)}$ has the effect of transposing A_p while leaving the remaining constituents unchanged. Each rotation operator is linear on the $n \times n$ matrices over $\mathbf{B}$ and their product is the transposition operator, $R : A \to A^t$.

Example 4.1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ be a matrix over $\mathbf{B}$. Then $mc(A) = 3$. But $R^{(1)}(A) = \sigma_1 A_1^t + \sigma_1^* A = A^t$, the transpose matrix of A, has Boolean maximal column rank 2. Consider $B = A \oplus 0_{n-3,n-3}$ for $n \geq 3$. By property (3.2), the rotation operator does not preserve Boolean maximal column rank 3 on $\mathbf{M}_{m,n}(\mathbf{B})$. □

Lemma 4.4 (**[5]**). *If T is a linear operator on the $m \times n$ matrices $(m, n \geq 1)$ over a general Boolean algebra $\mathbf{B}$, then the followings are equivalent.*

(1) T preserves Boolean ranks 1 and 2.

(2) T is in the group of operators generated by the congruence (if m=n , also the rotation) operators.

Theorem 4.1. *Suppose T is a linear operator on $\mathbf{M}_{m,n}(\mathbf{B})$ for $m \geq 2$ and $n \geq 1$. Then the following are equivalent.*

(1) T preserves Boolean maximal column rank.

(2) T preserves Boolean maximal column ranks 1,2 and 3.

(3) T is a congruence operator.

Proof. Clearly (1) implies (2). Now we show that (2) implies (3). Assume T preserves Boolean maximal column rank 1,2 and 3. Then by Lemma 4.2, each constituent T_p preserves binary Boolean maximal column ranks 1,2 and 3. For $A \in \mathbf{M}_{m,n}(\mathbf{B})$, Theorem 3.1 implies $b_1(A) = mc_1(A)$ for $b_1(A) \leq 2$. Thus T_p preserves binary Boolean ranks 1 and 2, and hence T preserves Boolean ranks 1 and 2 by Lemma 4.3. So T is in the group of operators generated by the congruence(if m=n, also the rotation) operators by Lemma 4.4. But the rotation operator does not preserve Boolean maximal column ranks 3 by Example 4.1. Hence T is a congruence operator since T preserves Boolean maximal column rank 3. That is (2) implies (3). Finally, assume that T is a congruence operator of the form $T(A) = UAV$, where U and V are invertible $m \times m$ and $n \times n$ Boolean matrices respectively. Then T preserves Boolean maximal column rank by Lemma 4.1. Hence (3) implies (1). □

If $m \leq 2$, then the linear operators that preserve maximal column rank on $\mathbf{M}_{m,n}(\mathbf{B})$ are the same as the Boolean rank-preservers, which were characterized in [5].

Thus we have characterizations of the linear operators that preserve the Boolean maximal column rank of general Boolean matrices.

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