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Linear operators that preserve column rank of fuzzy matrices

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Abstract: For each $m \geqslant 2$ and $n \geqslant 3$, we characterize the linear operators, T, on the $m \times n$ fuzzy matrices that preserve column rank. That is, T preserves column rank if and only if T strongly preserves column rank 1 and it preserves column rank 3. Other characterizations of column rank preserving operators are also given.

Keywords: Operators; fuzzy matrix; column rank.

1. Introduction

There are many papers on the study of linear operators that preserve the semiring rank of matrices over several semirings. Beasley and Pullman have obtained many results on them [1-3, and 5]. But there are few papers on column rank preservers of the matrices over semirings. Recently Beasley and Song obtained characterizations on column rank preservers of matrices over nonnegative integers in [6] and of Boolean matrices in [9]. Beasley and Pullman [3] obtained characterizations of rank preserving linear operators over fuzzy matrices.

In this paper, we study the extent to which known properties of linear operators preserving the ranks of matrices over 'chain semiring' (see Section 2) carry over to operators preserving column ranks. We obtain some characterizations of linear operators that preserve column rank of fuzzy matrices and of matrices over chain semirings which is a more general class than Boolean algebra.

2. Preliminaries

A *semiring* is a binary system $(\mathbb{S}, +, \times)$ such that $(\mathbb{S}, +)$ is an Abelian monoid (identity 0), $(\mathbb{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in $\mathbb{S}$ and $1 \neq 0$. Usually $\mathbb{S}$ denotes both the semiring and the set and $\times$ is denoted by juxtaposition.

Let the set of $m \times n$ matrices with entries in a semiring $\mathbb{S}$ be denoted by $\mathbf{M}_{m,n}(\mathbb{S})$. The $m \times n$ zero matrix $0_{m,n}$ and the $n \times n$ identity matrix I_n are defined as if $\mathbb{S}$ were a field. Addition, multiplication by scalars, and the product of matrices are also defined as if $\mathbb{S}$ were a field. Thus $\mathbf{M}_{m,m}(\mathbb{S})$ is a semiring under matrix addition and multiplication.

If V is a nonempty subset of $\mathbb{S}^k \equiv \mathbf{M}_{k,1}(\mathbb{S})$ that is closed under addition and multiplication by scalars, then V is called a *vector space* over $\mathbb{S}$. The notions of subspace and of spanning sets are the same as if $\mathbb{S}$ were a field.

As with fields, a *basis* for a vector space V is a spanning subset of least cardinality. That cardinality is

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the *dimension*, dim(V), of V. The *column space* of an $m \times n$ matrix A over $\mathbb{S}$ is the vector space spanned by its columns. The *column rank* $c(A) = c_{\mathbb{S}}(A)$ of an $m \times n$ matrix A over $\mathbb{S}$ is the dimension of the column space.

The *semiring rank* of a nonzero matrix A in $\mathbb{M}_{m,n}(\mathbb{S})$ is the least integer k such that $A = BC$ for some B in $\mathbb{M}_{m,k}(\mathbb{S})$ and some C in $\mathbb{M}_{k,n}(\mathbb{S})$. The rank of $0_{m,n}$ is 0. We denote the rank of A by $r(A)$ or by $r_{\mathbb{S}}(A)$.

It follows directly from the definitions that for all $m \times n$ matrices A over a semiring $\mathbb{S}$:

(2.1) $0 \leq r(A) \leq \min(m, n)$, and $0 \leq c(A) \leq n$;

(2.2) $r(B) \leq r(A)$ for all submatrices B of A;

(2.3) $r(A) = r(A')$, where A' is the transpose matrix of A.

Also these are known:

(2.4) $c(B) \leq c(A)$ if B is obtained by deleting some rows of A;

(2.5) The semiring rank of a nonzero matrix A is the minimum number of semiring rank 1 matrices which sum to A ([3, Lemma 2.1]);

(2.6) $r(A) \leq c(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$ ([4, Lemma 2.3]);

(2.7) $c(A) = \min\{r(X): AX = A\}$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$ ([4, Lemma 2.4]).

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally ordered by $<$, that is, $\mathbb{S}$ is a chain under $<$ (i.e. $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y$ as $\max(x, y)$ and xy as $\min(x, y)$ for all x, y in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound, then $\mathbb{S}$ becomes a semiring: a *chain semiring*.

Let $\mathbb{H}$ be any nonempty family of sets nested by inclusion, $0 = \bigcap_{x \in \mathbb{H}} x$, and $1 = \bigcup_{x \in \mathbb{H}} x$. Then $\mathbb{S} = \mathbb{H} \cup \{0, 1\}$ is a chain semiring.

Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S} = \{\beta \in \mathbb{R}: \alpha \leq \beta \leq \omega\}$. Then $\mathbb{S}$ is a chain semiring with α = '0' and ω = '1'. It is isomorphic to the chain semiring in the previous example with $\mathbb{H} = \{[\alpha, \beta]: \alpha \leq \beta \leq \omega\}$.

If in particular we choose the real numbers 0 and 1 as α and ω in the previous example, then the system of $m \times n$ matrices over $\{\beta \in \mathbb{R}: 0 \leq \beta \leq 1\}$ is the *fuzzy matrices*.

If we take $\mathbb{H}$ to be a singleton, say $\{a\}$, and denote empty subset by 0 and $\{a\}$ by 1, the resulting chain semiring (called a *Boolean algebra* $\mathbb{B}$) is a subsemiring of every chain semiring.

Beasley and Pullman [4] obtained the following relations between semiring rank and column rank over $\mathbb{M}_{m,n}(\mathbb{S})$.

Theorem 2.1 ([4, Theorem 2 and 3]). *Let $\mu(\mathbb{S}, m, n)$ be the largest k such that for all $m \times n$ matrices A over $\mathbb{S}$, $r(A) = c(A)$ if $r(A) \leq k$. Then*

(1) *for a chain semiring $\mathbb{K}$ other than the Boolean algebra $\mathbb{B}$, we have*

$$\mu(\mathbb{K}, m, n) = \begin{cases} 2 & \text{if } m \geq 2 \text{ and } n = 2, \\ 1 & \text{otherwise.} \end{cases}$$

(2) *for the Boolean algebra $\mathbb{B}$, we have*

$$\mu(\mathbb{B}, m, n) = \begin{cases} 1 & \text{whenever } \min(m, n) = 1, \\ 3 & \text{for all } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

Hereafter, otherwise specified, $\mathbb{K}$ will denote a chain semiring which is not the Boolean algebra $\mathbb{B}$, all matrices will denote the $m \times n$ matrices over a chain semiring and we will write $\mathbb{M}$ or $\mathbb{M}(\mathbb{K})$ for $\mathbb{M}_{m,n}(\mathbb{K})$. Column vectors (members of $\mathbb{M}_{k,1}$) will usually be denoted by boldface lower case letters such as $\mathbf{y}$.

Since 1 is the only invertible member of the multiplicative monoid of $\mathbb{K}$, the permutation matrices (obtained by permuting the columns if I_n) are the only invertible members of $\mathbb{M}_{n,n}(\mathbb{K})$.

Lemma 2.1. *The column rank of a matrix is unchanged by pre- or post-multiplication by an invertible matrix. Further, the column rank of a 2×2 matrix is unchanged by transposition.*

Proof. Let P, Q be fixed invertible matrices. For a matrix A, (2.7) implies that

$$\begin{aligned} c(PA) &= \min\{r(X): (PA)X = PA\} \\ &= \min\{r(X): AX = A\}, \qquad \text{since } P \text{ is invertible}, \\ &= c(A). \end{aligned}$$

Further Q is in fact a permutation matrix, so $c(AQ) = c(A)$. The rest follows from Theorem 2.1 using (2.3). □

A function T mapping $\mathbb{M}_{m,n}(\mathbb{S})$ into $\mathbb{M}_{m,n}(\mathbb{S})$ is called an *operator* on $\mathbb{M}_{m,n}(\mathbb{S})$. The operator T

(i) is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{S}$ and all $A, B \in \mathbb{M}_{m,n}(\mathbb{S})$,

(ii) *preserves semiring rank h* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $r(A) = h$, $r(T(A)) = r(A)$,

(iii) is a *congruence operator* if there exist invertible matrices U and V in $\mathbb{M}_{m,m}(\mathbb{S})$ and $\mathbb{M}_{n,n}(\mathbb{S})$ respectively such that $T(A) = UAV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$,

(iv) is a *transposition operator* if $m = n$ and $T(A) = A^t$ for all $A \in \mathbb{M}_{m,m}(\mathbb{S})$.

(v) *preserves column rank k* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $c(A) = k$, $c(T(A)) = c(A)$.

Lemma 2.2. *Congruence operators on $\mathbb{M}(\mathbb{K})$ are linear, are bijective, and preserve all column ranks and all semiring ranks.*

Proof. Linearity follows from the linearity of matrix multiplication. The rest follows from Lemma 2.1. □

Let j_k denote the column vector of length k all of whose entries are 1. When the orders are understood, we may drop the subscripts on j_k. Let $E_{i,j}$ be the $m \times n$ matrix all of whose entries are 0 except the (i, j)th, which is 1.

We define the *norm* of an arbitrary $m \times n$ matrix X by $\|X\| = j^t X j$ the sum of all entries in X. That is, $\|X\|$ is the maximum entry in X. Note the mapping $X \to \|X\|$ respects matrix addition and scalar multiplication.

Using Theorem 2.1, we can apply the results for semiring rank 1 or for semiring rank 2 over $m \times 2$ matrices to those for column rank 1 or for column rank 2 over $m \times 2$ matrices, respectively. Thus we obtain the following Lemma 2.3 by the analogue proofs of those in [3].

Lemma 2.3. *Suppose*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $c(A) = 2$ if and only if $ad \neq bc$.

Proof. See [3], Lemma 2.7. □

Lemma 2.4. *If H is a 2×2 submatrix of A, then $c(H) \leq c(A)$.*

Proof. If $c(H)=k$, $k=0$, 1 and 2 then $r(H)=k$ by Theorem 2.1. Hence $c(A)\geq r(A)\geq k$ by (2.6) and (2.2). □

Example 2.1. Consider

$$A=\begin{pmatrix}0&1&1&0&0\\1&0&0&1&0\\1&1&0&0&1\\1&1&1&1&0\end{pmatrix}$$

over $\mathbb{B}$, the two element Boolean algebra. Then $c_{\mathbb{B}}(A)=3$ because the last three columns are linearly independent and span the column space of A. If we delete column 5 from A to obtain a submatrix H of A, then $c_{\mathbb{B}}(H)=4$ since the four columns of H are linearly independent and span the column space of H.

Thus the column rank of an $h\times k$ submatrix may be greater than that of the given matrix when $h>2$ and $k>2$.

3. Linear operators that preserve column rank over $\mathbb{M}_{m,n}(\mathbb{K})$

Unless otherwise specified, all matrices in this section are $m\times n$ matrices over a chain semiring $\mathbb{K}$. Let $\mathbb{M}$ denote the set of all $m\times n$ matrices over $\mathbb{K}$. The set of matrices of column rank 1 over a fixed chain semiring $\mathbb{K}$ is denoted by C_1.

In the sequel, we shall adopt the convention $m\leq n$.

Column rank 1 matrices A, B are said to be *separable* if there is a matrix X with $c(X)=1$ such that either $1=c(A+X)<c(B+X)$ or $1=c(B+X)<c(A+X)$. The matrix X is said to *separate* A from B.

Using Theorem 2.1 we can apply some results in [3] for semiring rank 1 matrices to those for column rank 1 matrices. Thus we obtain the following Theorem 3.1 by the analogue proof of that in [3].

Theorem 3.1. *Distinct column rank* 1 *matrices are separable if and only if at least one of them is not a scalar multiple of J.*

Proof. See the proof of [3, Theorem 3.1]. □

The symbol $\leq$ is read entrywise, i.e., $X\geq Y$ if and only if $x_{ij}\leq y_{ij}$ for all (i,j).

Lemma 3.1 [3, Lemma 4.3]. *If T is a linear operator on $\mathbb{M}$, $\min(m,n)>1$, T preserves norm, and $A\leq T(A)$, then $T^q(A)=T^{mn-1}(A)$ for all $q\geq mn$.*

Lemma 3.2. *Let T be a linear operator on $\mathbb{M}$ with $\min(m,n)>1$. If T preserves norm and column rank* 1 *but is not injective on C_1, then T reduces the column rank of some matrix from k (≥ 2) to* 1.

Proof. Since T is not injective, $T(A)=T(B)$ for some A, B in C_1 with $A\neq B$. If $A=\alpha J$ and $B=\beta J$, then $\alpha=\beta$ because T preserves norms, contradicting our assumption that $A\neq B$. Therefore by Theorem 3.1 some column rank 1 matrix X separates A from B. Say $c(X+A)=1$ and $c(X+B)=k\geq 2$. Since

$$T(X+B)=T(X)+T(B)=T(X)+T(A)=T(X+A),$$

T reduces the column rank of $X + B$ from k to 1. □

We say that a linear operator T *strongly preserves column rank* 1 provided that $c_{\mathbb{S}}(T(X)) = 1$ if and only if $c_{\mathbb{S}}(X) = 1$ for $X \in \mathbb{M}_{m,n}(\mathbb{S})$.

Lemma 3.3. *If T is a linear operator on* $\mathbb{M}$, $\min(m, n) > 1$, *and T strongly preserves column rank* 1, *then T preserves norm.*

Proof. Let $A \in \mathbb{M}$, $\alpha = \|A\|$ and $\beta = \|T(A)\|$; then $A = \alpha A$ and $\beta = \|T(A)\| = \|T(\alpha A)\| = \alpha\, \|T(A)\| \leqslant \alpha$. Suppose $\beta < \alpha$. Then for some (i, j), $a_{ij} = \alpha$. Let Y be the matrix whose entries are all α except for $y_{ij} = 0$. Then $\alpha J = A + Y$. So $c(A + Y) = 1$. Since $c(\beta A + Y) \geqslant 2$ by Lemmas 2.3 and 2.4, and $c(\beta A + Y) \leqslant 2$ by construction, we have $c(\beta A + Y) = 2$. By the linearity of T and the definition of β, we have $T(\beta A) = \beta T(A) = T(A)$. Hence

$$T(\beta A + Y) = T(\beta A) + T(Y) = T(A) + T(Y) = T(A + Y) = \alpha T(J).$$

So T reduces the column rank of $\beta A + Y$ from 2 to 1, contrary to the hypothesis. Thus T preserves norm. □

Lemma 3.4. *Suppose T is a linear operator on* $\mathbb{M}$ *and* $\min(m, n) > 1$. *If T strongly preserves column rank* 1, *then T permutes Γ, where $\Gamma = \{E_{ij}: 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$.*

Proof. By Lemma 3.3, T preserves norm. Therefore by Lemma 3.2, T is injective on C_1, the $m \times n$ column rank 1 matrices over $\mathbb{S}$. Suppose $T(E_{pq})$ is not in Γ for some (p, q). Now $T(E_{pq}) = \sum \tau_{ij} E_{ij}$. But $\|T(E_{pq})\| = 1$, so $\tau_{uv} = 1$ for some (u, v). Without loss of generality, we may assume $(u, v) = (p, q)$, because if P, Q are permutation matrices, then the linear operator $X \to PT(X)Q$ preserves the column ranks T preserves (see Lemma 2.1) and permutes Γ if and only if T does. Let $E = E_{pq}$. Then $E \leqslant T(E)$, so $E \neq T(E) \leqslant T^2(E) \leqslant \cdots \leqslant T^k(E) = T^{k+h}(E)$, where k is the least integer for which equality holds and $h \geqslant 0$ is arbitrary. By Lemma 3.1 we are assured that k exists and is less than mn. Let $B = T^{k-1}(E)$. Then $B \neq T(B)$ but $T(B) = T(T(B))$, despite the fact that B, $T(B)$ are both in C_1 and T is injective on C_1. This contradiction implies that T maps Γ into Γ. By injectivity, T permutes Γ. □

Let $\mathbb{B}$ be the two element subsemiring $\{0, 1\}$ of $\mathbb{K}$, and α be a fixed member of $\mathbb{K}$, other than 1. For each x in $\mathbb{K}$ define $x^\alpha = 0$ if $x \leqslant \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}$. Its entrywise extension to a mapping $A \to A^\alpha$ of $\mathbb{M}(\mathbb{K})$ onto $\mathbb{M}(\mathbb{B})$ preserves matrix sums and products and multiplication by scalars. We call A^α the *α-pattern* of A.

Example 3.1. Let $\mathbb{K}$ be a chain semiring other than $\mathbb{B}$. For a nonzero nonunit $p \in \mathbb{K}$, consider

$$A = \begin{bmatrix} p & p & p \\ p & p & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then $c(A) = 3$, since all the three columns of A are in the set of spanning columns for the column space $\langle A \rangle$. But $c(A^t) = 2$. Consider $B = A \oplus 0_{m-3,m-3}$ for $m \geqslant 3$. If T is a transposition operator over $\mathbb{M}_{m,m}(\mathbb{K})$, then $T(B) = B^t$ has column rank 2 while $c(B) = 3$. Thus a transposition operator does not preserve column rank 3.

Earlier, the semiring rank preserving operators on $\mathbb{M}_{m,n}(\mathbb{K})$ or $M_{m,n}(\mathbb{B})$ were characterized in [3] or

[1] respectively. Also the column rank preserving operators on $\mathbb{M}_{m,n}(\mathbb{B})$ were characterized in [9]. For our purpose, we write those results as follows.

Lemma 3.5 (1) ([9, Theorem 2.2]). *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ for $n \geq m \geq 4$. Then T preserves column ranks 1, 2 and 3 if and only if it is a congruence operator. Moreover the transposition operator on $\mathbb{M}_{m,n}(\mathbb{B})$ does not preserve column rank 3 for $m = n \geq 4$.*

(2) ([1, Theorem 4.1, and Theorem 4.2]). *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ with $n \geq m > 1$. Then T preserves semiring ranks 1 and 2 if and only if it is in the group of operators generated by congruence and transposition operators.*

(3) ([3, Theorem 4.2]). *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $n \geq m > 1$. Then T is bijective and preserves semiring rank 1 if and only if it is in the group of operators generated by congruence and transposition operators.*

We say that an $m \times n$ matrix X is a *column matrix* if $X = \boldsymbol{x}(\boldsymbol{e}_i)^t$ for some $\boldsymbol{x} \in \mathbb{S}^m$ and $\boldsymbol{e}_i \in \mathbb{S}^n$, where $\boldsymbol{e}_i$ is the vector with 1 in the ith position and 0 elsewhere.

Theorem 3.2. *Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring $\mathbb{K}$, where $m \geq 2$ and $n \geq 3$. If T strongly preserves column rank 1, and it preserves column rank 3, then T is a congruence operator.*

Proof. Let $\bar{\mathbb{M}} = \mathbb{M}_{m,n}(\mathbb{B})$. Lemma 3.4 and linearity imply that T maps $\bar{\mathbb{M}}$ into itself. Let $\bar{T}$ denote the restriction of T to $\bar{\mathbb{M}}$. From the definition of column rank, the column rank $c_{\mathbb{B}}(X)$ of a member X of $\bar{\mathbb{M}}$ is at least $c_{\mathbb{K}}(X)$, its column rank as a member of $\mathbb{M}_{m,n}(\mathbb{K})$, because $\mathbb{B} \subset \mathbb{K}$. On the other hand, the mapping that takes a matrix A to its 0-pattern A^0 preserves matrix sums and multiplication by scalars. Hence $c_{\mathbb{B}}(X) = c_{\mathbb{K}}(X)$ for all X in $\bar{\mathbb{M}}$. Therefore $\bar{T}$ strongly preserves column rank 1, and it preserves column rank 3.

Case 1 ($n \geq m \geq 4$). Since $\bar{T}$ also permutes Γ by Lemma 3.4 and it strongly preserves column rank 1, $\bar{T}$ must map a column matrix to either a column matrix or transpose of a column matrix if $m = n \geq 4$. For the latter case, $\bar{T}$ is a composition of a transposition operator and pre-multiplication by a permutation matrix. Since transposition operator does not preserve column rank 3 by Lemma 3.5(1), $\bar{T}$ must map a column matrix to a column matrix. Thus the linearity of $\bar{T}$ implies that $c_{\mathbb{B}}(\bar{T}(X)) \leq c_{\mathbb{B}}(X)$ for all X in $\bar{\mathbb{M}}$. In particular, $\bar{T}$ preserves column rank 2. Hence $\bar{T}$ is a congruence operator on $\bar{\mathbb{M}}$ by Lemma 3.5(1). The corresponding matrices U, V are also invertible over $\mathbb{M}$; in fact, they are just permutation matrices. Let $A \in \mathbb{M}$. Then $T(A) = \sum a_{ij} T(E_{ij}) = \sum a_{ij} \bar{T}(E_{ij})$ as $E_{ij} \in \bar{\mathbb{M}}$. Since $\bar{T}(E_{ij}) = UE_{ij}V$ for all i, j, by definition of congruence operator, the result follows from the linearity of matrix multiplication.

Case 2 ($n = 3$ and $2 \leq m \leq 3$). For this case, $\bar{T}$ strongly preserves semiring rank 1 and it preserves semiring rank 3 by Theorem 2.1(2). If $r_{\mathbb{B}}(X) = 2$ for $X \in \bar{\mathbb{M}}$, then X can be factored as a sum of two matrices X_1 and X_2 whose semiring ranks are 1 by (2.5). Thus $\bar{T}(X_1) = \bar{T}(X_2) + \bar{T}(X)$ has semiring rank 2 or less. Since $\bar{T}$ strongly preserves semiring rank 1, $r_{\mathbb{B}}(\bar{T}(X)) = 2$. That is, $\bar{T}$ preserves semiring rank 2. Therefore $\bar{T}$ is in the group of operators generated by congruence (and if $m = n = 3$, also the transposition) operators by Lemma 3.5(2). Let $A \in \mathbb{M}$. Then $T(A) = \sum a_{ij} T(E_{ij}) = \sum a_{ij} \bar{T}(E_{ij})$ since each $E_{ij} \in \bar{\mathbb{M}}$. It follows that there are permutation matrices U and V ($m \times m$ and $n \times n$ respectively) such that in the case that $n = 3$ and $m = 2$, $T(A) = UAV$, while in the case $m = n = 3$, $T(A)$ is either UAV or UA^tV. However, since transposition operator does not preserve column rank 3 on $\mathbb{M}$ by Example 3.1, we see that in fact, T must be a congruence operator. □

Theorem 3.3. *Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring with $m \geq 2$ and $n \geq 3$. Then the following are equivalent:*

(i) *T preserves all column ranks.*

(ii) *T strongly preserves column rank 1 and it preserves column rank 3.*

(iii) *T is a congruence operator.*
(iv) *T is bijective and preserves column ranks* 1 *and* 3.

Proof. Theorem 3.2 establishes that (ii) implies (iii). According to Lemma 2.2, (iii) implies (i) and (iv). If T satisfies (iv), then T is in the group of operators generated by congruence and transposition operators by Lemma 3.5(3) and Theorem 2.1. Since the transposition operator does not preserve column rank 3, T must be a congruence operator. Therefore (iv) implies (iii). □

How necessary is it that $m \geqslant 2$ and $n \geqslant 3$? If $m = n = 2$, then a linear operator that preserves all column ranks is the same as a linear operator that preserves all semiring ranks by Theorem 2.1. The characterizations of the semiring rank preservers were obtained in [3].

Thus we have characterizations of the linear operators that preserve the column rank of matrices over a chain semiring (and in particular, of fuzzy matrices) when $m \geqslant 2$ and $n \geqslant 3$.

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