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ANALYSIS OF IDEMPOTENT MATRICES OVER NONNEGATIVE INTEGERS

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Abstract

An $n \times n$ matrix A is called *idempotent* if $A^2 = A$. Analogues of characterizations of types of idempotent binary Boolean matrices are determined for the semiring of nonnegative integers. Consequently we obtain that a nonnegative integer matrix A is idempotent if and only if it is a sum of pure rectangle parts and line parts.

Keywords: Nonnegative integers, idempotent matrix, frame, rectangle part, line part.

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1 Introduction

A semiring is essentially a ring in which only the zero is required to have an additive inverse. The set of all nonnegative integers, the Boolean algebra of subsets of a finite set, and the fuzzy set are combinatorially interesting examples of semirings.

The characterization of idempotent matrices in abstract algebraic systems is a vital problem that is crucial for the understanding the structure of these systems and in many other applications ([4, 6, 8]). It is well-known that over any field the structure of idempotent matrices is very simple, that is, each idempotent matrix is similar to a diagonal matrix with 0 and 1 on the main diagonal. But for matrices over algebraic systems that are not fields, this problem is far from being solved yet.

Bapat et al. ([1]) obtained characterizations of nonnegative real idempotent matrices by some techniques and, Song and Kang ([9]) characterized all idempotent binary Boolean matrices that are sums of four cells. Recently, Beasley et al. ([3]) showed that a binary Boolean matrix is idempotent if and only if it can be expressed as a sum of line parts and rectangle parts of certain specific structure.

In this paper, we extend the results for the binary Boolean algebra to semiring of all nonnegative integers.

2 Preliminaries and definitions

DEFINITION 2.1. ([5, 6]) A *semiring* $\mathbb{S}$ consists of a set $\mathbb{S}$ and two binary operations, addition $+$, and multiplication $\cdot$, such that

(1) $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);
(2) $\mathbb{S}$ is a monoid under multiplication (identity denoted by 1);
(3) multiplication is distributive over addition on both sides;
(4) $s0 = 0s = 0$ for all $s \in \mathbb{S}$.

A semiring $\mathbb{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Then $\mathbb{Z}_+$ is a commutative antinegative semiring which has no zero-divisors.

Let $\mathbf{B} \equiv \mathbf{B}_k$ be the (*general*) *Boolean algebra* of subsets of a k element set S_k and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of S_k. Union is denoted by $+$, and intersection by $\cdot$; 0 denote the null set and 1 the set S_k. Under these two operations, $\mathbf{B}$ is a commutative antinegative semiring; all of its elements, except 0 and 1, zero-divisors. In particular, $\mathbf{B}_1 = \{0, 1\}$ is called the *binary Boolean algebra*.

Let $\mathbf{F} = [0, 1]$ be the set of reals between 0 and 1 with addition $(+)$, multiplication $(\cdot)$ and the ordinary order $\leq$ such that $x + y = \max\{x, y\}$ and $x \cdot y = \min\{x, y\}$ for all $x, y \in \mathbf{F}$. Then $\mathbf{F}$ becomes a commutative antinegative semiring which has no zero-divisors, and called the *fuzzy set*.

Throughout this paper, we will assume that $\mathbb{S}$ is a commutative antinegative semiring, and let $\mathcal{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{S}$. The usual definitions for addition, multiplication by scalars, and the product of matrices over fields are applied to $\mathbb{S}$ as well. The zero matrix is denoted by O_n, the identity matrix by I_n and the matrix with all entries equal to 1 is denoted by J_n. The matrix in $\mathcal{M}_n(\mathbb{S})$ all of whose entries are zero except its $(i,j)^{\text{th}}$, which is 1, is denoted by E_{ij}. We call this a *cell*.

The following is an immediate consequence of the rules of matrix multiplication.

PROPOSITION 2.2. *For any cells E_{ij} and E_{uv}, we have $E_{ij}E_{uv} = E_{iv}$ or O_n according as $j = u$ or $j \neq u$.*

DEFINITION 2.3. A matrix $A \in \mathcal{M}_n(\mathbb{S})$ is called *idempotent* if $A^2 = A$.

The matrices O_n and I_n are clearly idempotents in $\mathcal{M}_n(\mathbb{S})$. By Proposition 2.2, we have that all diagonal cells are idempotents, but all off-diagonal cells are not idempotents. The matrix J_n is idempotent over the general Boolean algebra, while it is not idempotent over the nonnegative integers because ${J_n}^2 = nJ_n$ in $\mathcal{M}_n(\mathbb{Z}_+)$.

DEFINITION 2.4. Let $A = [a_{ij}]$ be a matrix in $\mathcal{M}_n(\mathbb{S})$. If $a_{ij} \neq 0$ for some i and j, then A_{ij} is denoted by $A_{ij} = a_{ij}E_{ij}$, and it is said to be the $(i,j)^{\text{th}}$ *weighted cell in* A. When $i \neq j$, we say that A_{ij} is *off-diagonal*; A_{ii} is *diagonal*.

For a matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{S})$, A can be written uniquely as $\sum_{i,j=1}^{n} a_{ij}E_{ij}$. Thus the matrix A is a sum of $(i,j)^{\text{th}}$ weighted cells in A for all $i,j = 1,\ldots,n$.

We say that a matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{S})$ *dominates* a matrix $B = [b_{ij}] \in \mathcal{M}_n(\mathbb{S})$ if and only if $b_{ij} \neq 0$ implies that $a_{ij} \neq 0$, and we write $A \sqsupseteq B$ or $B \sqsubseteq A$.

Let A, B, C and D be matrices in $\mathcal{M}_n(\mathbb{S})$. Then we can easily show that

$$\text{if } A \sqsubseteq B \text{ and } C \sqsubseteq D, \text{ then we have } AC \sqsubseteq BD. \tag{2.1}$$

Let $A = [a_{i,j}]$ be a matrix in $\mathcal{M}_n(\mathbb{S})$. For any cell E_{ij}, we have $E_{ij} \sqsubseteq A$ if and only if $a_{ij} \neq 0$ if and only if A_{ij} is the $(i,j)^{\text{th}}$ weighted cell in A if and only if $A_{ij} \sqsubseteq A$.

LEMMA 2.5. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. For $m \geq 2$, if $A_{i_1j_1}, A_{i_2j_2}, \ldots, A_{i_mj_m}$ are $(i_r, j_r)^{\text{th}}$ weighted cells in A for $r = 1, \ldots, m$, then $A_{i_1j_1}A_{i_2j_2}\cdots A_{i_mj_m} \sqsubseteq A$.*

Proof. Since each $A_{i_rj_r}$ is a $(i_r, j_r)^{\text{th}}$ weighted cell in A, we have $A_{i_rj_r} \sqsubseteq A$ for $r = 1, \ldots, m$. It follows from (2.1) that $A_{i_1j_1}A_{i_2j_2}\cdots A_{i_mj_m} \sqsubseteq A^m$. Since A is idempotent, we have $A^m = A$ for $m \geq 2$. Thus the result follows. ∎

LEMMA 2.6. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If F is an off-diagonal cell with $F \sqsubseteq A$, then there exist distinct cells G and H with $G, H \sqsubseteq A$ such that $F = GH$. Moreover if both cells G and H are off-diagonal, then the cells F, G and H are mutually distinct.*

Proof. Let $A_{i_1j_1}, A_{i_2j_2}, \ldots, A_{i_mj_m}$ be $(i_r, j_r)^{\text{th}}$ weighted cells in A for $r = 1, \ldots, m$. Then $A = \sum_{r=1}^{m} A_{i_rj_r}$. Since A is idempotent, we have

$$\sum_{r=1}^{m} {A_{i_rj_r}}^2 + \sum_{s,t=1, s\neq t}^{m} A_{i_sj_s}A_{i_tj_t} = A^2 = A = \sum_{r=1}^{m} A_{i_rj_r}.$$

Since $F \sqsubseteq A$, we have either $F \sqsubseteq {A_{i_rj_r}}^2$ or $F \sqsubseteq A_{i_sj_s}A_{i_tj_t}$ for some $r, s, t \in \{1, \ldots, m\}$ with $s \neq t$. Since F is off-diagonal, it follows from Proposition 2.2 that $F \not\sqsubseteq {A_{i_rj_r}}^2$. Thus we have $F \sqsubseteq A_{i_sj_s}A_{i_tj_t}$ for some $s, t \in \{1, \ldots, m\}$ with $s \neq t$. Let G and H be cells with $A_{i_sj_s} \sqsubseteq G$ and $A_{i_tj_t} \sqsubseteq H$. Then clearly $G, H \sqsubseteq A$ and $F \sqsubseteq GH$ so that $F = GH$ since F and GH are all cells. Furthermore, if G and H are off-diagonal, then F, G and H are mutually distinct by Proposition 2.2. ∎

DEFINITION 2.7. Let $A_{i_1j_1}, A_{i_2j_2}, A_{i_3j_3}$ and $A_{i_4j_4}$ be four weighted cells in $A \in \mathcal{M}_n(\mathbb{S})$. Then $X = \sum_{r=1}^{4} A_{i_rj_r}$ is called a *frame in A* if the four nonzero entries of X constitute a rectangle with at least one entry on diagonal; X is *pure* if it has only one nonzero diagonal entry.

For example, consider a matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ 0 & a_6 & a_7 \end{bmatrix} \in \mathcal{M}_3(\mathbb{S})$, where $a_i \neq 0$ for all $i = 1, \ldots, 7$. Then A has just 2 frames, and they are

$$X_1 = \begin{bmatrix} a_1 & a_2 & 0 \\ a_4 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 0 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & a_6 & a_7 \end{bmatrix}.$$

Here X_2 is pure, while X_1 is not.

Let $A \in \mathcal{M}_n(\mathbb{S})$ be a given matrix. For $i = 1, \ldots, n$, we define an i^{th} *row matrix* $\boldsymbol{R}_i(A)$ of A as a matrix whose i^{th} row is the same as the i^{th} row of A and the other rows are zero. Similarly, we can define a j^{th} *column matrix* $\boldsymbol{C}_j(A)$ of A for $j = 1, \ldots, n$. If the matrix A is clear from the context, we write $\boldsymbol{R}_i(A)$ and $\boldsymbol{C}_j(A)$ as $\boldsymbol{R}_i$ and $\boldsymbol{C}_j$, respectively. Thus we have

$$A = \sum_{i=1}^{n} \boldsymbol{R}_i(\mathrm{A}) = \sum_{j=1}^{n} \boldsymbol{C}_j(A) \quad \text{or} \quad A = \sum_{i=1}^{n} \boldsymbol{R}_i = \sum_{j=1}^{n} \boldsymbol{C}_j.$$

DEFINITION 2.8. Let A be a matrix in $\mathcal{M}_n(\mathbb{S})$. Then $RP(i)[A] \in \mathcal{M}_n(\mathbb{S})$ is called an i^{th} *rectangle part* of A if the following hold:

(1) there is a frame X in A such that $A_{ii} \sqsubseteq X$;

(2) for any $1 \le l, k \le n$, if A_{li} and A_{ik} are weighted cells in A, then A_{lk} is a weighted cell in A;

(3) $RP(i)[A]$ is the matrix with the smallest number of weighted cells in A, and dominates all frames in A dominating A_{ii}.

Suppose that $A \in \mathcal{M}_n(\mathbb{S})$ has the i^{th} rectangle part $RP(i)[A]$. Let

$$\{A_{j_1 i}, \ldots, A_{j_s i}\} \quad \text{and} \quad \{A_{i i_1}, \ldots, A_{i i_t}\}$$

be the sets of all off-diagonal weighted cells in $\boldsymbol{C}_i$ and $\boldsymbol{R}_i$, respectively. Then we have

$$RP(i)[A] = A_{ii} + \sum_{k=1}^{s} A_{j_k i} + \sum_{l=1}^{t} A_{i i_l} + \sum_{k=1}^{s} \sum_{l=1}^{t} A_{j_k i_l}.$$

Let $B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_5 & 0 & b_6 \\ b_7 & b_8 & b_9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, where all b_i's are nonzero in $\mathbb{S}$. Then there exists the 3^{th} rectangle part of B and $RP(3)[B] = B_{11} + B_{12} + B_{13} + B_{31} + B_{32} + B_{33}$. However any i^{th} rectangle part of B does not exist for all $i = 1, 2$ and 4.

DEFINITION 2.9. Let A be a matrix in $\mathcal{M}_n(\mathbb{S})$. Then $LP(i)[A] \in \mathcal{M}_n(\mathbb{S})$ is called an i^{th} *line part* of A if the following hold:

(1) $A_{ii} \sqsubseteq A$ and $LP(i)[A] = \boldsymbol{R_i} + \boldsymbol{C_i}$;

(2) $\boldsymbol{R_i} + \boldsymbol{C_i}$ is the i^{th} row matrix or the i^{th} column matrix of A.

Let $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $Y = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, where a, b and c are nonzero in $\mathbb{S}$. Then $LP(1)[X] = aE_{11} + bE_{12}$ and $LP(2)[X] = bE_{12} + cE_{22}$, while Y do not have line parts.

3 Some results

In this section, we give some properties of idempotent matrices in $\mathcal{M}_n(\mathbb{S})$, where $\mathbb{S}$ is a commutative antinegative semiring. For this purpose, we shall analyze the structures of the sums of weighted cells.

For any matrix $A = [a_{ij}]$ in $\mathcal{M}_n(\mathbb{S})$, define the matrix $A^* = [a_{ij}^*]$ in $\mathcal{M}_n(\mathbb{B}_1)$ as $a_{ij}^* = 1$ if and only if $a_{ij} \neq 0$. If $\mathbb{S}$ is a semiring which has no zero-divisors, then we can easily show that

$$(A + B)^* = A^* + B^*, \quad (AB)^* = A^*B^* \quad \text{and} \quad (\alpha A)^* = \alpha^* A^* \qquad (3.1)$$

for all $A, B \in \mathcal{M}_n(\mathbb{S})$ and for all $\alpha \in \mathbb{S}$.

The following is an immediate consequence of (3.1).

PROPOSITION 3.1. *Let $\mathbb{S}$ be a semiring which has no zero-divisors. If A is idempotent in $\mathcal{M}_n(\mathbb{S})$, then A^* is idempotent in $\mathcal{M}_n(\mathbb{B}_1)$.*

For weighted cells $A_1, A_2, \ldots, A_m$ in $A \in \mathcal{M}_n(\mathbb{S})$, they are called *collinear* if $\sum_{i=1}^{m} A_i \sqsubseteq X$, where X is either an i^{th} row matrix or a j^{th} column matrix of A.

LEMMA 3.2. ([3]) *Let A be a nonzero matrix in $\mathcal{M}_n(\mathbb{B}_1)$.*

(1) *If all cells in A are off-diagonal, then A is not idempotent;*

(2) *Assume there exists an off-diagonal cell $F \sqsubseteq A$ such that for any diagonal cell $E \sqsubseteq A$, E and F are not collinear. If A is idempotent, then F is in a pure frame in A.*

COROLLARY 3.3. *Let $A_1, \ldots, A_m$ be all weighted cells in $A \in \mathcal{M}_n(\mathbb{S})$.*

(1) *If all A_i are diagonal, then A is idempotent if and only if all A_i are idempotent;*

(2) *If $\mathbb{S}$ has no zero-divisors and all A_i are off-diagonal, then A is not idempotent.*

Proof. Let $A = \sum_{i=1}^{m} A_i$. (1) Suppose that all A_i are diagonal. It follows from Proposition 2.2 that A is idempotent if and only if

$$A_1A_1 + \cdots + A_mA_m = A^2 = A = A_1 + \cdots + A_m$$

if and only if ${A_i}^2 = A_i$ for all $i = 1, \ldots, m$.

(2) If all A_i are off-diagonal, then by (3.1) A^* is just sum of off-diagonal cells in $\mathcal{M}_n(\mathbb{B}_1)$. It follows from Lemma 3.2-(1) that A^* is not idempotent. Therefore A is not idempotent by Proposition 3.1. ■

COROLLARY 3.4. *Let $\mathbb{S}$ be a semiring which has no zero-divisors, and let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If A has an off-diagonal weighted cell A_{ij} such that A_{ij} is not collinear with any diagonal weighted cell in A, then A_{ij} is in a pure frame in A.*

Proof. Since A is idempotent in $\mathcal{M}_n(\mathbb{S})$ with $A_{ij} \sqsubseteq A$, so is A^* in $\mathcal{M}_n(\mathbb{B}_1)$ with $E_{ij} \sqsubseteq A^*$ by Proposition 3.1. It follows from Lemma 3.2-(2) that E_{ij} is in a pure frame in A^*, equivalently A_{ij} is in a pure frame in A. ■

Let $A \in \mathcal{M}_n(\mathbb{S})$. For $1 \le i,j \le n$, $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are said to be *(i,j)-disjoint* if $A_{ix}A_{yi} = O_n$ for any off-diagonal weighted cell A_{ix} in $\boldsymbol{R_i}$ and for any off-diagonal weighted cell A_{yi} in $\boldsymbol{C_j}$.

LEMMA 3.5. *Let A be idempotent in $\mathcal{M}_n(\mathbb{S})$. If $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint, then A_{ij} is the weighted cell in A.*

Proof. If $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint, then there exist off-diagonal weighted cells A_{ix} in $\boldsymbol{R_i}$ and A_{yj} in $\boldsymbol{C_j}$ such that $A_{ix}A_{yj} \neq O_n$. Since A is idempotent, $A_{ix}A_{yj} \sqsubseteq A$ by Lemma 2.5. It follows from Proposition 2.2 that $x = y$ so that $a_{ij} \neq 0$. Hence A_{ij} is the weighted cell in A. ■

The number of nonzero entries of $A \in \mathcal{M}_n(\mathbb{S})$ is denoted by $|A|$.

LEMMA 3.6. *Let $\mathbb{S}$ be a semiring which has no zero-divisors, and let A be idempotent in $\mathcal{M}_n(\mathbb{S})$ with $A_{ii} \sqsubseteq A$ for some i. If $|\boldsymbol{R_i}| = s+1$ and $|\boldsymbol{C_i}| = t+1$, then there exist exactly $s \cdot t$ frames in A dominating A_{ii}.*

Proof. If $s = 0$ or $t = 0$, then the result is straightforward. Thus we can assume that $s, t \ge 1$. Since A is idempotent, Lemma 2.5 and Proposition 2.2 implies that for any off-diagonal weighted cells

$$A_{ki} \sqsubseteq \boldsymbol{R_i} \sqsubseteq A \quad \text{and} \quad A_{il} \sqsubseteq \boldsymbol{C_i} \sqsubseteq A,$$

their product $A_{ki}A_{il} \sqsubseteq A$ so that $A_{kl} \sqsubseteq A$. Therefore, the four weighted cells A_{ii}, A_{ki}, A_{il} and A_{kl} are in a frame in A for each k, l such that $A_{ki} \sqsubseteq A$ and $A_{il} \sqsubseteq A$. Thus A has at least $s \cdot t$ frames such that each frame dominates A_{ii}. It follows from the definition of frame that A has at most $s \cdot t$ frames dominating A_{ii}. ■

Let $\mathbb{B} = \mathbb{B}_2$ be the Boolean algebra of a two element set S_2, and let

$$A = \begin{bmatrix} 1 & \sigma_1 & \sigma_2 \\ \sigma_2 & 0 & \sigma_2 \\ \sigma_1 & \sigma_1 & 0 \end{bmatrix} \in \mathcal{M}_3(\mathbb{B}_2). \tag{3.2}$$

Then we can easily show that A is idempotent in $\mathcal{M}_3(\mathbb{B}_2)$. Notice that $|\boldsymbol{R}_1| = 2+1 = |\boldsymbol{C}_1|$. But A has only two frames dominating A_{11}. Thus, the condition that $\mathbb{S}$ has no zero-divisors in Lemma 3.6 is needed.

Let $A = [a_{ij}]$ be idempotent in $\mathcal{M}_n(\mathbb{S})$, where $\mathbb{S}$ is a semiring which has no zero-divisors. If $a_{ii} \neq 0$, $|\boldsymbol{R}_1| > 1$ and $|\boldsymbol{C}_1| > 1$, then Lemma 3.6 shows that the i^{th} rectangle part of A exists.

THEOREM 3.7. *Let $\mathbb{S}$ be a semiring which has no zero-divisors. If A is idempotent in $\mathcal{M}_n(\mathbb{S})$, then every weighted cell in A is in either a rectangle part or a line part of A.*

Proof. It follows directly from Corollary 3.4 and Lemma 3.6. ■

The matrix A in (3.2) also shows that the condition($\mathbb{S}$ has no zero-divisors) is needed in Theorem 3.7 because A has neither a rectangle part nor a line part.

4 Idempotent matrices over nonnegative integers

In this Section, we shall characterize idempotent matrices over nonnegative integers.

Let A be a nonzero idempotent matrix in $\mathcal{M}_n(\mathbb{Z}_+)$. Then A has at least one diagonal weighted cell in A by Corollary 3.3-(2). Furthermore we can easily show that if A_{ii} is a diagonal weighted cell in A, then we have $A_{ii} = E_{ii}$.

LEMMA 4.1. *Let A_{ij} be an off-diagonal weighted cell in $A \in \mathcal{M}_n(\mathbb{Z}_+)$. If A_{ii} and A_{jj} are diagonal weighted cells in A, then A is not idempotent.*

Proof. Since A_{ii}, A_{jj} and A_{ij} are weighted cells in A, we have that a_{ii}, a_{jj} and a_{ij} are all nonzero in $\mathbb{Z}_+$. Then the $(i,j)^{\text{th}}$ entry b_{ij} of A^2 is greater than that of A because

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} \geq a_{ii}a_{ij} + a_{ij}a_{jj} = (a_{ii} + a_{jj})a_{ij} \geq 2a_{ij} > a_{ij}.$$

Hence A is not idempotent. ∎

Let $RP(i)[A]$ be an i^{th} rectangle part of $A \in \mathcal{M}_n(\mathbb{S})$. Then $RP(i)[A]$ is called *pure* if it has only one nonzero diagonal entry.

COROLLARY 4.2. *If $RP(i)[A]$ is an i^{th} rectangle part of an idempotent matrix $A \in \mathcal{M}_n(\mathbb{Z}_+)$, then it is pure.*

Proof. It follows from Lemma 4.1. ∎

LEMMA 4.3. *Let A be a matrix in $\mathcal{M}_n(\mathbb{Z}_+)$ with $A_{ii} \sqsubseteq A$ and $A_{jj} \sqsubseteq A$ for some indices i and j. If $\boldsymbol{R}_i$ and $\boldsymbol{C}_j$ are not (i,j)-disjoint, then A is not idempotent.*

Proof. If $i \neq j$, the result follows from Lemmas 3.5 and 4.1. So we may assume that $i = j$. Since $\boldsymbol{R}_i$ and $\boldsymbol{C}_i$ are not (i,i)-disjoint, there exist at least two off-diagonal weighted cells $A_{ix} \sqsubseteq \boldsymbol{R}_i$ and $A_{yi} \sqsubseteq \boldsymbol{C}_i$ such that $A_{ix}A_{yi} \neq O_n$. By Proposition 2.2, we have $x = y$. Since $A_{xi} \sqsubseteq A$ and $A_{ix} \sqsubseteq A$, it follows from Lemma 2.5 that $A_{xi}A_{ix} \sqsubseteq A$ and hence $A_{xx} \sqsubseteq A$ by Proposition 2.2. That is, $A_{ix}, A_{ii}, A_{xx} \sqsubseteq A$. By Lemma 4.4, A is not idempotent. ∎

Consider a matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_4(\mathbb{Z}_+).$$

Then A is the sum of one 1^{st} pure rectangle part and one 4^{th} line part. But $\boldsymbol{R}_1$ and $\boldsymbol{C}_4$ are not $(1,4)$-disjoint. By Lemma 4.3, A is not idempotent.

THEOREM 4.4. *Let A be a matrix in $\mathcal{M}_n(\mathbb{Z}_+)$. Then A is idempotent if and only if the followings are satisfied:*

(1) *there exist integers $s, t \geq 0$ such that A is the sum of s pure rectangle parts and t line parts,*

(2) *each pure rectangle part is idempotent,*

(3) *each line part is idempotent,*

(4) *for any* $i, j \in \{1, \ldots, n\}$, $\boldsymbol{R}_i$ *and* $\boldsymbol{C}_j$ *are* (i, j)*-disjoint.*

Proof. It is routine to check that a matrix satisfying the four conditions is idempotent. To show the opposite implication, assume that A is idempotent. Let A_{ij} be a weighted cell in A. By Theorem 3.7 and Corollary 4.2, A_{ij} is in some pure rectangle part or some line part of A. Thus, there exist integers $s, t \geq 0$ such that A is the sum of s pure rectangle parts and t line parts. Thus (1) is satisfied. (4) follows from Lemma 4.1. (2) and (3) are obvious by (4). ■

Thus we have characterizations of all idempotent matrices over the nonnegative integers.

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