A Transformation in the Product of Wiener Spaces

Jin-oh Hyun · Bong-soo Ko

直積위너空間의 變換

玄 進 五 · 高 鳳 秀

Summary

In this paper we extend Bearman's results, rotations in the product of two Wiener spaces, and give several results which prove useful in dealing with transformations in Wiener space.

1. Introduction

Let T=[0,1] and let $C_0(T)$ denote Wiener space, that is, the space of real-valued continuous functions on T which vanish at t=0, Let $0=t_0<t_1<\ldots<t_n=1$ and let $-\infty\leqslant\alpha_i\leqslant\beta_i\leqslant\infty$, i = 1,2,...,n. Subsets of $C_0(T)$ of the type

$$I=\left\{x\in C_0(T):\alpha_i<x(t_i)\leqslant\beta_i,\ i=1,2,\ldots,n\right\}$$

are called intervals. We denote the class of all intervals $\mathfrak{L}$. It can be shown that $\mathfrak{L}$ is semi-algebra. Now we defined a set function m on $\mathfrak{L}$ as follows;

$$m_1(I)=\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_n}^{\beta_n}W_n(\vec{t},\vec{u})\,du_1\ldots du_n$$

where $\vec{t}=(t_1,t_2,\ldots,t_n)$, $\vec{u}=(u_1,u_2,\ldots,u_n)$ and

$$W(\vec{t},\vec{u})=\left\{(2\pi)^n\prod_{i=1}^{n}(t_i-t_{i-1})\right\}^{-1/2}\exp\left\{\sum_{i=1}^{n}-\frac{(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\right\},\ (u_0=0).$$

m_1 is countably additive on $\mathfrak{L}$ and can be extended in the usual way to the σ-algebra $\sigma(\mathfrak{L})$ generated by the intervals and then can be further extended so as to be a complete measure. This completed measure space is denoted by $(C_0(T),\mathcal{S}_1,m_1)$ and $\mathcal{S}_1$ is called the class of Wiener measurable sets.

For $x\in C_0(T)$, let $\|x\|=\max_{t\in[0,1]}|x(t)|$. Then $(C_0(T),\|\cdot\|)$ is a separable Banach space.

Let $\mathcal{B}$ be the collection of all sets of the form $J_{\vec{t}}(B)$ for all $\vec{t}$ and all Borel sets B in R^n. Then $\mathcal{B}$ is an algebra of subsets of $C_0(T)$. Let $\sigma(\mathcal{B})$ be the σ-algebra generated by $\mathcal{B}$ and $\mathcal{B}(C_0(T))$ be the class of Borel sets in $C_0(T)$. Then it is well known that $\sigma(\mathfrak{L})=\sigma(\mathcal{B})=\mathcal{B}(C_0(T))$.

In [1] Bearman obtained the results; Let R denote the linear transformation from the plane to the plane which rotates each vector through an angle θ. Let

$$R_\theta(u,v) = (U,V)$$

where $U = u\cos\theta - v\sin\theta$, $V = u\sin\theta + v\cos\theta$. Define $R_\theta^* : C_0(T) \times C_0(T) \to C_0(T) \times C_0(T)$ to be $R_\theta^*(x, y) = (X, Y)$ by $R_\theta(x(t), y(t))$. Then

$$m_1 \times m_1 = (m_1 \times m_1) \cdot (R_\theta^*)^{-1}$$

on $\mathcal{B}(C_0(T) \times C_0(T))$.

In this paper we extend Bearman's results, and give several results which prove useful in dealing with transformations in Wiener space.

2. Transformations in the Product of Wiener Spaces.

Let $u_1, u_2, .., u_n$ and $u_1^*, u_2^*, .., u_n^*$ be any two systems of Cartesian coordinates. Let

$$v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n$$
$$= v_1^* e_1^* + v_2^* e_2^* + \ldots + v_n^* e_n^*$$

be the representations of a given vector v in these two coordinate sstems; here $e_1, \ldots, e_n$ and $e_1^*, \ldots, e_n^*$ are unit vectors in the positive $u_1, \ldots, u_n$ and $u_1^*, \ldots, u_n^*$ directions respectively. We adopt the notation

$$e_i^* \cdot e_j = a_{ij} \qquad (i,j = 1,2,\ldots,n)$$

Then we have

$$v_i^* = \sum_{j=1}^{n} a_{ij} v_j \qquad (i = 1,2, \ldots, n)$$

A similar consideration leads to the inverse formulas

$$v_j = \sum_{i=1}^{n} a_{ij} v_j^* \qquad (j = 1, 2, \ldots, n)$$

Furthermore

$$\sum_{i=1}^{n} a_{ij} a_{im} = \begin{cases} 0 & (j \neq m) \\ 1 & (j = m) \end{cases}$$

If both coordinate systems under consideration are right-handed, then the determinant,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{mn} \end{vmatrix} = 1$$

Let $T: R^n \to R^n$ to be $T(v_1, \ldots, v_n) = (V_1, \ldots, V_n)$, where $V_i = \sum_{j=1}^{n} a_{ij} v_j$, $i = 1,2, \ldots, n$. Then T and T^{-1} preserve Euclidean distance in R^n and inner product as well as Lebesgue measure.

<u>Theorem 2-1.</u> If $T^*: C_0(T)^n \to C_0(T)^n$ to be $(X_1, \ldots, X_n) = T^*(x_1, \ldots, x_n)$ by

$$X_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) \quad t \in [0, 1]$$

and $i = 1,2, \ldots, n$, then $\prod^{n} m_1 = (\prod^{n} m_1)(T^*)^{-1}$ on $\mathcal{B}(C_0(T)^n)$.

<u>Proof.</u> Since the intervals generate $\mathcal{B}(C_0(T))$, the set of the form $I_1 \times \ldots \times I_n$ generate $\mathcal{B}(C_0(T))^n$. We may assume that $I_1, I_2, \ldots, I_n$ are based on the same restriction points. Let

$$\prod_{i=1}^{n} I_i = \Big\{ (x_1, \ldots, x_n) : \alpha_i < x(t) \leq \beta_i, \ldots, \psi_i < x(t) \leq \omega_i, i=1,2,\ldots, m \Big\}$$

Then

$$(\prod^{n} m_1)(\prod_{i=1}^{n} I_i) = m_1(I_1) m_1(I_2) \ldots m_1(I_n)$$
$$= \left\{ [(2\pi)^m \prod_{i=1}^{n} (t_i - t_{i-1})]^{-1/2} \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} \exp\Big(\sum_{i=1}^{m} - \frac{(v_i^1 - v_{i-1}^1)^2}{2(t_i - t_{i-1})} \Big) dv_1^1 \ldots dv_m^1 \right\} \ldots$$
$$\left\{ [(2\pi)^m \prod_{i=1}^{m} (t_i - t_{i-1})]^{-1/2} \int_{\psi_1}^{\omega_1} \ldots \int_{\psi_m}^{\omega_m} \exp\Big(\sum_{i=1}^{m} - \frac{(v_i^n - v_{i-1}^n)^2}{2(t_i - t_{i-1})} \Big) dv_1^n \ldots dv_m^n \right\}$$

$$= [(2\pi)^m \prod_{i=1}^{m} (t_i - t_{i-1})]^{-\frac{n}{2}} \int_{\alpha_1}^{\beta_1} \cdots \int_{\psi_1}^{\omega_1} \cdots \int_{\alpha_m}^{\beta_m} \cdots \int_{\psi_m}^{\omega_m} \exp \sum_{j=1}^{n} \sum_{i=1}^{m} - \frac{(V_i^j - V_{i-1}^j)^2}{2(t_i - t_{i-1})} dV_1^1 \cdots dV_1^n \ldots dV_1^n \ldots dV_m^1 \ldots dV_m^n,$$

by the change by variables sending $(v_i^1, \ldots, v_i^n)$ to $(V_i^1, \ldots, V_i^n) = T(v_i^1, \ldots, v_i^n)$, $i = 1, 2, \ldots, m$, and the Fubini theorem, and hence

$$\prod^{n} m_1 = (\prod^{n} m_1)(T^*)^{-1}$$

on $\mathcal{B}(C_0(T))^n$.

The next result follows immediately from Theorem 2-1 and the integral transport formula [3].

__Corollary 2-2.__ $F(x_1, \ldots, x_n)$ is measurable on $(C_0(T)^n \mathcal{S}_1^n)$ if and only if $F(T^*(x_1, \ldots, x_n))$ is measurable on $(C_0(T)^n, \mathcal{S}_1^n)$ and in this case, we get

$$\int_{C_0(T)^n} F(x_1, \ldots, x_n)\, d(\prod^{n} m_1)(x_1, \ldots, x_n)$$

$$= \int_{C_0(T)^n} F(T^*(x_1, \ldots, x_n))\, d(\prod^{n} m_1)(x_1, \ldots, x_n)$$

$$= \int_{C_0(T)^n} F(\sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j)\, d(\prod^{n} m_1)(x_1, \ldots, x_n).$$

__Corollary 2-3.__ φ is a measurable function on $C_0(T)$ if and only if $\varphi(\sum_{j=1}^{n} a_{ij} x_j)$ is measurable on $(C_0(T)^n, \mathcal{S}_1^n)$ for some i and we have

$$\int_{C_0(T)} \varphi(X_j) dm_1(X_j) = \int_{C_0(T)^n} \varphi(\sum_{j=1}^{n} a_{ij} x_j)(x_1, \ldots, x_n).$$

__Prof.__ Let $F(X_1, \ldots, X_n) = \varphi(X_i)$ for some i $(1 \leq i \leq n)$, Then

$$\int_{C_0(T)} \varphi(X_i) dm_1(X_i) = \int_{C_0(T)^n} F(X_1, \ldots, X_n)\, d(\prod^{n} m_1)(X_1, \ldots, X_n)$$

$$= \int_{C_0(T)^n} F(\sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j)\, d(\prod^{n} m_1)(x_1, \ldots, x_n)$$

$$= \int_{C_0(T)^n} \varphi(\sum_{j=1}^{n} a_{ij} x_j)\, d(\prod^{n} m_1)(x_1, \ldots, x_n)$$

__Corollary 2-4.__ Let $P_1, P_2, \ldots, P_n$ be positive real numbers. Then $\varphi(\sqrt{P_1^2 + \ldots + P_n^2}\, \omega)$ is Wiener measurable as a function of ω if and only if $\varphi(\sum_{i=1}^{n} P_i x)$ is measurable on $(Co(T)^n, \mathcal{S}_1^n)$ and in this case

$$\int_{C_0(T)} \varphi(\sqrt{\sum_{i=1}^{n} P_i^2}\, \omega)\, dm_1(\omega) = \int_{C_0(T)^n} \varphi(\sum_{i=1}^{n} p_i x_i)\, d(\prod^{n} m_1)(x_1, \ldots, x_n).$$

__Proof.__ Let

$$a_{i1} = \frac{P_2}{\sqrt{\sum_{k=1}^{n} P_k^2}}, \ldots, a_{in} = \frac{P_n}{\sqrt{\sum_{k=1}^{n} P_k^2}}$$

in Corollary 2-3. Then

$$\int_{C_0(T)} \varphi(\sqrt{\sum_{k=1}^{n} P_k^2}\, X_i)\, dm_1(X_i) = \int_{C_0(T)^n} \varphi(\sum_{j=1}^{n} P_j x_j)\, d(\prod^{n} m_1)(x_1, \ldots, x_n)$$

Let σ_n be the partition $0 = t_0 < t_1 < \ldots < t_{2^n} = 1$ where $t_k = \frac{k}{2^n}$ for $k = 0, 1, 2, \ldots, 2^n$. Given x in $C_0(T)$, let $S\sigma_n(x) = \sum_{k=1}^{2^n} [x(t_k) - x(t_{k-1})]^2$ For $\lambda > 0$, let

$$C_\lambda \equiv \left\{ x \text{ in } C_0(T) : \lim_{n \to \infty} S\sigma_n(x) = \lambda^2 \right\}$$

and let

$$D \equiv \left\{ x \text{ in } C_0(T) : \lim_{n\to\infty} S\sigma_n(x) \text{ fails to exist} \right\}$$

Note that $\lambda C_\mu = C_{\lambda\mu}$

Let m_λ be the Borel measure given by

$$m_\lambda(B) = m_1(\lambda^{-1} B)$$

for B in $\mathcal{B}(C_0(T))$. Since $m_1(C_1)=1$ and $\lambda^{-1} C_\lambda = C_1$, we see that m_λ is concentrated on the Borel set C_λ: i.e. $m_\lambda(C_\lambda)=1$. Let $\mathcal{S}_\lambda$ denote the σ-algebra obtained by completing $(C_0(T), \mathcal{B}(C_0(T)), m_\lambda)$.

Theorem 2-5. Let $P_1, \dots, P_n$ be positive numbers. The following assertions are equivalent:

(a) $f(\sqrt{\sum_{i=1}^n P_i^2}\, Z)$ is an m_1-measurable function of Z

(b) $f(Z)$ is an $m_{\sqrt{\sum_{i=1}^n P_i^2}}$ measurable function of Z

(c) $f(\sum_{i=1}^n x_i)$ is an $m_{P_1} \times \dots \times m_{P_n}$-measurable function of $x_1, \dots, x_n$.

(d) $f(\sum_{i=1}^n P_i x_i)$ is an $m_1 \times \dots \times m_1$-measurable function of $x_1, \dots, x_n$.

If any one (and hence all) of (a)-(d) holds, then

$$\int_{C_0(T)} f(\sqrt{\textstyle\sum_{i=1}^n P_i^2}\, Z)\, dm_1(Z) \stackrel{*}{=}$$

$$= \int_{C_0(T)} f(Z)\, dm_{\sqrt{\sum_{i=1}^n P_i^2}}(Z)$$

$$= \int_{C_0(T)^n} f(\sum_{i=1}^n x_i)\, d(\prod_{i=1}^n m_{P_i})(x_1, \dots, x_n)$$

$$= \int_{C_0(T)^n} f(\sum_{i=1}^n P_i x_i)\, d(\prod m_1)(x_1, \dots, x_n)$$

where be $\stackrel{*}{=}$ we mean that if either side exists, both sides exist and they are equal

Proof. (a) ⇔ (b)

Consider $T : (C_0(T), \mathcal{S}_1, m_1) \to (C_0(T), \mathcal{S}_{\sqrt{\sum_{i=1}^n P_i^2}}, m_{\sqrt{\sum_{i=1}^n P_i^2}})$ by $T(Z) = \sqrt{\sum_{i=1}^n P_i^2}\, Z$.

Then T is a measurable transformation. For any real α,

$$(f\circ T)^{-1}(\alpha, \infty) = \frac{1}{\sqrt{\sum_{i=1}^n P_i^2}} f^{-1}(\alpha, \infty) \in \mathcal{S}_1$$

if and only if $f^{-1}(\alpha, \infty) \in \mathcal{S}$

$$\int_{C_0(T)} f(Z)\, dm_{\sqrt{\sum_{i=1}^n P_i^2}}(Z) = \int_{C_0(T)} (f\circ T)(Z)\, dm_1(Z)$$

$$= \int_{C_0(T)} f(\sqrt{\textstyle\sum_{i=1}^n P_i^2}\, Z)\, dm_1(Z)$$

(c) ⇔ (d). Consider $\varphi : C_0(T)^n \to C_0(T)$ (by $\varphi(x_1, \dots, x_n) = \sum_{i=1}^n x_i$)

and

$$T: (C_0(T)^n, \mathcal{S}_1^n, \prod^n m_1) \to (C_0(T)^n, \prod_{i=1}^n \mathcal{S}_{P_i}, \prod_{i=1}^n m_{P_i})$$

(by $T(x_1, \dots, x_n) = (p_1 x_1, \dots, p_n x_n)$). Then φ is continuous, and T is measurable. For any real α,

$$(f\circ\varphi\circ T)^{-1}(\alpha, \infty)\, T^{-1}((f\circ\varphi)^{-1}(\alpha, \infty)) \in \mathcal{S}_1^m$$

if any only if

$$(f\cdot\varphi)^{-1}(\alpha, \infty) \in \prod_{i=1}^n \mathcal{S}_{P_i}$$

$$\int_{C_0(T)^n} f(\sum_{i=1}^n x_i)\, d(\prod_{i=1}^n m_{P_i})(x_1, \dots, x_n)$$

$$= \int_{C_0(T)^n} (f\circ\varphi)(x_1, \dots, x_n)\, d(\prod^n m_1)\, T^{-1}(x_1, \dots, x_n)$$

$$= \int_{C_0(T)^n} f(\sum_{i=1}^n P_i x_i)\, d(\prod^n m_1)(x_1, \dots, x_n)$$

(a) ⇔ (d). By Corollary 2-4.

References

[1] J.E. Bearman, Rotations in the Product of two Wiener spaces, Proc. Amer. Math. Soc., 3, 1952.

[2] P.R. Halmos, Measure Theory, Springer-Verlag 1978.

[3] K. Jacobs. Measure and Integral. Academic Press, 1978.

[4] G.W. Johnson and D.L. Skoug, Scale-invariant measurability in Wiener space, Pacific J. of Math., 83, 1979.

[5] Proceedings of Symposia in pure and applied and mathematics. Vol 1, the workshop conference board, Korea.

국 문 초 록

直積위너空間의 變換

특수한 直積위너空間의 變換을 일반적인 直積위너空間으로 확장시키고, 그 變換에서 파생되는 결과들을 얻는다.