# The Extension Properties of Representations on the Lie Algebras

*Ko Bong-soo*

## Lie Algebras 상에서 정의된 Representations의 확장성에 관한 소고

高 鳳 秀

## 1. Introduction

General definition of a Lie ring is a ring L, except that there is no assumption of associativity or of a substitute for associativity, satisfying the following two axioms (for all a, b, c, e in L): (1) $a^2=0$ (anti-commutativity). (2) $ab.c+bc.a+ca.b=0$ (Jacobi identity). The definition of an algebra over a field F is a ring A which is simultaneously a vector space over F in such a way as to satisfy $\lambda .ab=\lambda a.b=a.\lambda b$ for all $\lambda$ in F and a, b in A. A Lie ring which is simultaneously an algebra is called a Lie algebra.

There is as yet not much of a coherent theory of general Lie rings or infinite dimensional Lie algebras. In this paper, the most part is discussing finite-dimensional Lie algebras, but of course there is no point in making this assumption when it is not needed. So there will be a blanket assumption of finite-dimensionality.

The basic example of a Lie ring is obtained as follows. Let A be any associative ring and introduce in A new multiplication $[ab]=ab-ba$; this operation we call commutation. The importance of this example is so immense that in the literature on Lie algebras it is customary to use brackets for the operation, even when the object under discussion is an abstract Lie ring and no associative ring is in sight.

A derivation of a ring A is an additive mapping D of A into itself satisfying $(ab)D=aD.b+a.bD$ for all $a, b \in A$. The product of two derivations is useless in this paper. But the commutator of two derivations turns out to be a derivation. Hence the derivations of any ring form a Lie ring. The property of the derivation of the ring A is similar to the one of the tangent vector on a manifold. In this motivation, some algebraic properties (e.g., power series with matrix coefficients) on Lie algebra could be applied to characterize the property of the tangent vector on a manifold.

師範大學 專任講師

The purpose of this paper, therefore, describes the extension properties of representatins on Lie algebras of power series rings over the commutative rings.

## 2. Basic Facts.

### Determination of the Lie Algebras of Low Dimensionalities.

The proof of following facts are in the N. Jacobson book [1962]

If $\{e_1, e_2, \cdots, e_n\}$ is a basis for a Lie algebra L, then $[e_i e_i]=0$ and $[e_i e_j]=-[e_j e_i]$. Hence in giving the multiplicatin table for the basis, it suffices to give the product $[e_i e_j]$ for $i<j$.

(A) dim L=1. Clearly, L is abelian (in Lie algebra version all Lie product of two elements is zero).

(B) dim L=2. Let $\{e, f\}$ be the basis of L.

(Subcase 1). If $L^2=0$, then L is abelian.

(Subcase 2). Let $L^2\neq 0$. Let $\{e, f\}$ be the basis of L. Then $[e\ f]=e=-[f\ e]$

(C) dim L=3. Let $\{e, f, g\}$ be the basis of L.

(Subcase 1). If $L^2=0$, then L is abelian

(Subcase 2). Let dim $L^2=1$ and $L^2$ be contained in the center of L. Then the multiplicable table is. (if assume $[f\ g]=e$) $[f\ g]=e$, $[e\ f]=0$, and $[e\ g]=0$.

(Subcase 3). Let dim $L^2=1$ and $L^2$ be not contained in the center of L. If we assume $[e\ f]\neq 0$, then the multiplicable table is:

$[e\ f]=e$, $[e\ g]=0$ and $[f\ g]=0$

(Sucase 4). Let dim $L^2=2$. Then $L^2$ cannot be the non-abelian two-dimensional Lie algebra. Let $L^2=Ke+Kf$, where K is the field. Then the multiplicable table is

$[e\ f]=0$, $[e\ g]=\alpha e+\beta f$ and $[f\ g]=\gamma e+\delta f$, where $\alpha, \beta, \gamma, \delta \in K$ and $\alpha\delta-\alpha\beta\neq 0$.

## 3. The Extension Properties.

Let K be an algebraically closed field, Mn(K) the ring of n x n matrices over K, and $K[t]=$

$$\{\sum_{i=0}^{\infty} a_i t^i \mid a_i \in K\}$$

the power series ring. Consider the set

$$M_n(K[t])=\{\sum_{i=0}^{\infty} A_i t^i \mid A_i \in M_n(K)\}$$

with the usual addition, multiplication and scalar multiplication over K, it is an algebra. Using the commutation:

$$[\sum_{i=0}^{\infty} A_i t^i \sum_{j=0}^{\infty} B_j t^j]=(\sum_{i=0}^{\infty} A_i t^i)(\sum_{i=0}^{\infty} B_i t^i)-(\sum_{j=0}^{\infty} B_j t^j)(\sum_{j=0}^{\infty} A_j t^j)$$

Mn(K[t]) becomes the Lie algebra. We also consider the sub-Lie algebra $L\infty=$

$$\{\sum_{i=1}^{\infty} A_i t^i \mid A_i \in M_n(K)\}=tM_n(K[t])$$

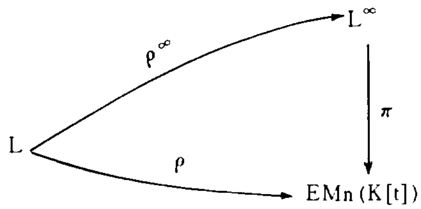
of Mn(K[t]). That L∞ is important may be equal to have the similar properties of tangent space on a manifold. We can thing another sub-Lie algebra

$$t^2 M_n(K[t])=\{\sum_{i=2}^{\infty} A_i t^i \mid A_i \in M_n(K)\}$$

on Mn(K[t]). Clearly, $t^2$Mn (K[t]) is a sub-Lie alebra of L∞. Let EMn (K[t]) be the quotient of $L^{\infty}$ under $t^2$Mn (K[t]). Then EMn (K[t]) becomes a Lie algebra and abelian.

### The Main Question

Let L be a Lie algebra, and let $\rho$ be a representation from L into EMn (K[t]). When can $\rho$ be extended to the representation $\rho^{\infty}$ which maps L to L∞, such that $\pi\rho^{\infty}=\rho$, where $\pi$ is the natural homomorphism from $L\infty$ into EMn (K[t]). i. e.,




The following result is the criterian for the extension of the representation $\rho$ on the abelian Lie algebra.

Theorem 1. Let $\{x^1, x^2 \cdot\cdot\cdot, x^n\}$ be the basis of L. and let

$$\rho(X^i) = \overline{\sum_{j=1}^{\infty} X_j^i t^j}, \ (1 \leqq i \leqq n)$$

the bar means the equivalence class for $\sum_{j=1}^{\infty} X_j^i t^j$ in $L^\infty$ Then $\rho$ can be extended to $\rho^\infty$ if and only if the subset $\{x^1{}_1, x^2{}_1, \cdot\cdot\cdot x^n{}_1\}$ of Mn(K) is commutative with respect to the matrix multiplication.

Proof. Suppose that $\rho$ can be extended to $\rho^\infty$. i.e. $\rho = \pi\rho^\infty$. Let $\rho^\infty(X^i) = \sum_{j=1}^{\infty} Y_j^i t^j$ for $1 \leq i \leq n$. Then $\pi\rho^\infty(X^i) = \rho(X^i)$ and so

$$\overline{\sum_{j=1}^{\infty} X_j^i t^j} = \overline{\sum_{j=1}^{\infty} Y_j^i t^j} \quad \text{for } 1 \leq i \leq n.$$

Since $\sum_{j=1}^{\infty} X_j^i t^j - \sum_{j=1}^{\infty} Y_j^i t^j \in t^2 M_n(K[t])$,

so $X_1^i = Y_1^i$ for.

$1 \leq i \leq n$. Therefore, $\rho^\infty(x^i) = X_1^i t + \sum_{j=2}^{\infty} Y_j^i t^j$.

To show that the set $\{x^1{}_1, x^2{}_1, \cdot\cdot\cdot x^n{}_1\}$ is commutative, we choose any two matrices $x^p{}_1$ and $x^q{}_1$ in the set. Since L is abelian, $[X^pX^q]=0$. By the hypothesis, $\rho^\infty$ is a Lie homomorphism, and so we note

$$0 = \rho^\infty([X^pX^q]) = [\rho^\infty(X^p)\, \rho^\infty(X^q)]$$
$$= \rho^\infty(X^p)\, \rho^\infty(X^q) - \rho^\infty(X^q)\, \rho^\infty(X^p).$$

By computing the product of power series, we obtain

$$0 = (X_1^p X_1^q - X_1^q X_1^p)\, t^2 + \sum_{j=3}^{\infty} Z_j t^j,$$

where $Z_j \in Mn(K)$. Hence $X_1^p X_1^q = X_1^q X_1^p$ i.e. commutative.

On the other hand, suppose that the set $\{x^1{}_1, x^2{}_1, \cdot\cdot\cdot x^n{}_1\}$ is commutative under the matrix multiplication. Let us define $\rho^\infty$ by $\rho^\infty(x^i) = x^i{}_1 t$ for $1 \leq i \leq n$. Then we can extend linearly $\rho^\infty$ as a function on L, and then clearly $\pi\rho^\infty = \rho$. To show that $\rho^\infty$ is an extension of $\rho$, we must prove that $\rho^\infty$ is a Lie homomorphism. Clearly the addition is preserved by $\rho^\infty$. so we need to check the preserving of the Lie product about $\rho^\infty$ Let $X, Y \in L$. and let.

$$X = \sum_{i=1}^{n} a^i X^i \quad \text{and} \quad Y = \sum_{i=1}^{n} b^i X^i,$$

where $a^i, b^i \in K$ (field). Since L is abelian, so $[XY]=0$, and so $\rho^\infty([XY])=0$ by the definition of $\rho^\infty$ Now

$$[\rho^\infty(X)\rho^\infty(Y)] = \rho^\infty(X)\rho^\infty(Y) - \rho^\infty(Y)\rho^\infty(X)$$
$$= \sum_{j=1}^{n} (\sum_{i=1}^{n} a^i b^j X_1^i X_1^j) t^2$$
$$- \sum_{i=1}^{n} (\sum_{j=1}^{n} b^j a^i X_1^j X_1^i) t^2.$$

Since $\{x^1{}_1, x^2{}_1, \cdot\cdot\cdot x^n{}_1\}$ is a commutative set with respect to the matrix multiplicatin, so $[\rho^\infty(X)\ \rho^\infty(Y)]=0$. Therefore, $\rho^\infty([XY]) = [\rho^\infty(X)\ \rho^\infty(Y)]$; i.e., $\rho^\infty$ is a Lie homomorphsm from L into $L^\infty$ which satisfies $\pi\rho^\infty = \rho$.

The following theorem is the criterian for the extension of the representation $\rho$ on the non-abelian Lie algebra with 2 dimensions.

Theorem 2. Let L be non-abelian Lie algebra and 2 dimensional, and let $\{X, Y\}$ be the basis of L such that $[XY]=Y$. Then in order to exist an extention $\rho\infty$ of $\rho$, the sufficient and necessary condition is that $\rho(Y)=\bar{0}$ in EMn(K[t]).

Proof. Suppose that there is an extension $\rho^\infty$ of $\rho$, and let

$$\rho(X) = \overline{\sum_{i=1}^{\infty} X_i t^i}, \quad \rho(Y) = \overline{\sum_{i=1}^{\infty} Y_i t^i}$$

and

$$\rho^\infty(X) = \sum_{i=1}^{\infty} A_i t^i, \quad \rho^\infty(Y) = \sum_{i=1}^{\infty} B_i t^i$$

Since $\pi \rho^\infty = \rho$, as the proof of Theorem 1, we can conclude that

$$\rho^\infty(X) = X_1 t + \sum_{i=2}^{\infty} A_i t^i$$

and

$$\rho^\infty(Y) = Y_1 t + \sum_{i=2}^{\infty} B_i t^i$$

Since $\rho^\infty(Y) = \rho^\infty([XY]) = [\rho^\infty(X)\rho^\infty(Y)]$

$$= \rho^\infty(X)\rho^\infty(Y) - \rho^\infty(Y)\rho^\infty(X),$$

$$Y_1 t + \sum_{i=2}^{\infty} B_i t^i = (X_1Y_1 - Y_1X_1)t^2$$

$$+ (X_1B_2 + X_2Y_1 - Y_1X_2 - B_2X_1)t^3 + \cdots$$

and so $Y_1 = 0$. Hence

$$\rho(Y) = \overline{\sum_{i=2}^{\infty} Y_i t^i} = \bar{0} \quad \text{in} \quad EM_n(k[t])$$

On the other hand, suppose $\rho(Y) = \bar{0}$ in EMn(K[t]), and let $\rho(X) = \overline{\sum_{i=1}^{\infty} X_i t^i}$. If we define $\rho^\infty$ by $\rho^\infty(X) = X_1 t$ and $\rho^\infty(Y) = 0$, then it becomes immediately a function on L, since L has the basis {X, Y}, and also $\pi \rho^\infty = \rho$. Since $\rho^\infty(Y) = \rho^\infty([XY]) = 0$ and $\rho^\infty(X)\ \rho^\infty(Y) - \rho^\infty(Y)\rho^\infty(X) = 0$, so the function $\rho^\infty$ is clearly a Lie homomorphism on L.

In Theorem 2, if we take the Lie algebra which satisfies $[XY] = X$, then we can replace $\rho(Y) = \bar{0}$ by $\rho(X) = \bar{0}$.

The following result is the necessary and sufficient condition for the extension of the representation on the non abelian Lie algebra with 3 dimensions. The result also has immediately corollaries on certain condition of the Lie algebra.

Theorem 3. Let L be a 3-dimensional Lie algebra with the basis {X, Y, Z} which satisfies $[XY] = Z$, $[XZ] = [YZ] = 0$. Then there is the extension $\rho^\infty$ of $\rho$ if and only if $\rho(Z) = \bar{0}$, $X_1Y_1 - Y_1X_1 = C$, $X_1C = CX_1$, and $Y_1C = CY_1$ for some C in Mn(K). Where

$$\rho(X) = \overline{\sum_{i=1}^{\infty} X_i t^i}, \quad \rho(Y) = \overline{\sum_{i=1}^{\infty} Y_i t^i},$$

$$\text{and} \quad \rho(Z) = \overline{\sum_{i=1}^{\infty} Z_i t^i}$$

Proof. Suppose that there is an extension $\rho^\infty$ of $\rho$ such that $\pi \rho^\infty = \rho$. Let

$$\rho^\infty(X) = \sum_{i=1}^{\infty} A_i t^i, \quad \rho^\infty(Y) = \sum_{i=1}^{\infty} B_i t^i,$$

$$\text{and} \quad \rho^\infty(Z) = \sum_{i=1}^{\infty} C_i t^i$$

We can conclude that $X_1 = A_1$, $Y_1 = B_1$, and $Z_1 = C_1$, since $\pi \rho^\infty = \rho$,

$$\rho^\infty(X) = X_1 t + \sum_{i=2}^{\infty} A_i t^i, \ \rho^\infty(Y) = Y_1 t + \sum_{i=2}^{\infty} B_i t^i$$

and

$$\rho^\infty(Z) = Z_1 t + \sum_{i=2}^{\infty} C_i t^i$$

Since $\rho^\infty$ is a Lie homomorphism, we can obtain $\rho^\infty(Z) = \rho^\infty([XY]) = [\rho^\infty(X), \ \rho^\infty(Y)] = \rho^\infty(X)\ \rho^\infty(Y) - \rho^\infty(Y)\ \rho^\infty(X)$.

And so

$$Z_1 t + \sum_{i=2}^{\infty} C_i t^i = (X_1Y_1 - Y_1X_1)t^2 + \cdots\cdots$$

Hence, $Z_1 = 0$ and $C_2 = X_1Y_1 - Y_1X_1$. This implies that

$$\rho(Z) = \overline{\sum_{i=2}^{\infty} Z_i t^i} = \bar{0}.$$

To show that $X_1C_2 = C_2X_1$ and $Y_1C_2 = C_2Y_1$, we first obtain the following identities:

$$0 = \rho^\infty([XY]) = \rho^\infty(X)\rho^\infty(Z) - \rho^\infty(Z)\rho^\infty(X)$$

$$= (X_1C_1 - C_1X_1)t^3 + \cdots$$

and, similary,

$$0 = \rho^\infty([YZ]) = (Y_1C_2 - C_2Y_1)t^3 \cdots$$

On the other hand, suppose that $\rho(Z) = \bar{0}$ and there is the matrix $C \in Mn(K)$ such that

$X_1Y_1-Y_1X_1=C,\ X_1C=CX_1,\ Y_1C=CY_1.$

Define $\rho^{\infty}(X)=X_1t,\ \rho^{\infty}(Y)=Y_1t$, and $\rho^{\infty}(Z)=(X_1Y_1-Y_1X_1)\ t^3$, then, clearly $\pi\rho\infty(Y)=\rho(Y)$, and $\pi\rho^{\infty}(Z)=\rho(Z)=\bar{0}$. We can also linearly extend $\rho^{\infty}$ on the whole L. Then $\pi\rho^{\cdot}=\rho$ and $\rho^{\cdot}$ is linear on L. To show $\rho^{\infty}$ is a Lie homomorphism from L into $L^{\infty}$, we choose $A, B\in L$ such that

$A=a_1X+a_2Y+a_3Z$ and $B=b_1X+b_2Y+b_3Z$,

where $a_i, b_i \in K$ $(i=1, 2, 3)$. Since L has the anti-commutative property,

$[AB]=a_1b_2[XY]+a_1b_3[XZ]+a_2b_1[YX]+a_3b_2[ZY]+a_1b_3[ZX]+a_2b_3[YZ].$

Since $[XY]=Z$ and $[XZ]=[YZ]=0$, we have the following equality:

$[AB]=(a_1b_2-a_2b_1)\ Z.$

Hence $\rho^{\infty}\ ([AB])=(a_1b_2-a_2b_1)\ \rho^{\infty}(Z)=(a_1b_1-a_2b_1)(X_1Y_1-Y_1X_1)t^2.$

By computation we get the following result:

$\rho^{\infty}(A)\ \rho^{\infty}(B)-\rho^{\infty}(B)\ \rho^{\infty}(A)$

$=a_1b_2\ (X_1Y_1-Y_1X_1)t^3+a_1b_3[X_1(X_1Y_1-Y_1X_1)X_1]t^3$

$+a_2b_1[Y_1X_1-X_1Y_1]t^2+a_2b_3[Y_1(X_1Y_1-Y_1X_1)-(X_1Y_1-Y_1X_1)Y_1]t^3$

$+a_3b_1[(X_1Y_1-Y_1X_1)X_1-X_1(X_1Y_1-Y_1X_1]t^3$

Claim:

$X_1(X_1Y_1-Y_1X_1)-(X_1Y_1-Y_1X_1)X_1=0$ and

$Y_1(X_1Y_1-Y_1X_1)-(X_1Y_1-Y_1X_1)Y_1=0$

If we assume that the claim is true, then

$\rho([AB])=\rho^{\infty}(A)\ \rho^{\infty}(B)-\rho^{\infty}(B)\ \rho^{\infty}(A).$

This implies that $\rho^{\infty}$ is a Lie homomorphism.

To prove the above claim, let us consider the following identity:

(*) $(X_1Y_1X_1-Y_1X_1X_1)-(X_1X_1Y_1-X_1Y_1X_1)+(Y_1X_1X_1-X_1Y_1X_1)-(X_1Y_1X_1-X_1X_1Y_1)=0$

Since $X_1Y_1-Y_1X_1=C$, we get two identities,

$X_1Y_1X_1-Y_1X_1X_1=CX_1$ and $X_1X_1Y_1-X_1Y_1X_1=X_1C$

Hence, the equation (*) changes as

$CX_1-X_1C+(Y_1X_1X_1-X_1Y_1X_1)-(X_1Y_1X_1-X_1X_1Y_1)=0$

Since $CX_1=X_1C$ by hypothesis, we have

$X_1(X_1Y_1-Y_1X_1)-(X_1Y_1-Y_1X_1)X_1=0$

Similarly, using $CY_1=Y_1C$, we can show the other's identity. This completes the proof.

Corollary 1. Let L have the properties as in Theorem 3, except $[XY]=X$, $[XZ]=[YZ]=0$. There is the extension $\rho^{\infty}$ of $\rho$ if and only if $\rho(X)=\bar{0}$ and $Y_1Z_1=Z_1Y_1$.

Proof. The method of the proof is almost all the same as the one of Theorem 3, except taking $\rho^{\cdot}(X)=0$, $\rho^{\cdot}(Y)=Y_1t$, and $k\rho^{\infty}(Z)=Z_1t$.

Corollary 2. Let L have the properties as in Theorem 3, except $[XY]=0$, $[XZ]=\alpha X+\beta Y$ and $[ZY]=\gamma X+\delta Y$, where $\alpha\delta-\gamma\beta\neq 0$. Then there is the extension $\rho^{\infty}$ of $\rho$ if and only if $\rho(X)=\bar{0}=\rho(Y)$.

Proof The method of the proof is also similar to the one of Theorem 3.

In the case of the Lie algebra being Nilpotent, I am trying to solve the extension problem, but I do not have any concrete solution of the problem. I have only partial solution of it. The solution is that: Let L be a n-dimensional Nilpotent Lie algebra with maximal nilpotent index m and with the basis $\{X, Y, \cdots, Z\}$. Let

$$\rho(X)=\overline{\sum_{i=1}^{\infty} X_i t^i},\quad \rho(Y)=\overline{\sum_{i=1}^{\infty} Y_i t^i},\ \ldots\ldots,$$

$$\rho(Z)=\overline{\sum_{i=1}^{\infty} Z_i t^i}$$

If $\rho$ can be extended to $\rho^{\infty}$, then any m-times Lie product of elements of $\{X_1, Y_1, \cdots\cdots, Z_1\}$ is zero. I am not sure whether the converse is true or not.

## Literature Cited

Chow, Y. 1978. *General Theory of Lie Algebras, I.* Gordon and Breach, New York.

Goto, M. 1978. *Semisimple Lie Algebras,* Marcel Dekker, Inc., New York.

Hochschild, G. P. 1981. *Basic Theory of Algebraic Groups and Lie Alebras.* Springer–Verlag.

Humphreys, J. E. 1972. *Intro. to Lie Algebras and Representation Theory,* Springer–Verlag.

Jacobson, N. 1971. *Exceptional Lie Algebras,* Marcel Dekker, Inc.

Jacobson, N. 1962. *Lie Algebras,* Dover Publications, Inc., New York.

Kaplansky, I. 1971. *Lie Algebras and Locally Compact Groups,* The University of Chicage Press.

Seligman, G. B. 1976. *Rational Methods in Lie Algebras,* Marcel Dekker.

Wan, Z. X. 1975. *Lie Algebras,* Pergamon Press, Oxford.

## 국 문 초 록

이 논문의 기본내용은 행렬들을 원소로 갖는 가환상에서 정의된 멱급수환에 대한 Lie Algebra의 representations의 확장 가능성을 논함.