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Hierarchical Optimal Control for River Pollution Control

*Kyung-Youn Kim**

강의 수질오염 제어를 위한 계층적 최적제어

김 경 연*

Summary

A discrete state space model for a multiple-reach river system is described using the dynamics of biochemical oxygen demand (BOD) and dissolved oxygen (DO). A hierarchical optimization technique, which is applicable to large-scale systems with time-delays in states, is also described to preserve stream quality in a river based on the interaction prediction method. the steady state tracking error of the proposed method is determined analytically and a necessary and sufficient condition on which a constant target tracking problem has zero steady-state error is derived. Computer simulations for the river pollution model illustrate the algorithm.

Introduction

In recent years there has been an increasing interest in the modeling and control of water quality in a river. Many parameters can be used to represent water quality in a stream, but it is widely known that the BOD and DO concentrations are the most universally accepted criteria (Haimes and Macko, 1973; Singh, 1975). In particular, the dynamics of DO concentration is dependent on that of BOD concentration. If the DO falls below certain levels or the BOD rises above certain levels, ecological balance of the river is often broken down. Therefore, it is necessary to control the BOD and DO levels to fluctuate between predefined bands while at the same time minimizing the cost of treatment in an optimal manner.

The state space model for the river with many polluters becomes large-scale time-delay systems (LSTD) (Singh *et. al*, 1981). A considerable research has been done on the optimal control of time-delay (TD) systems. They can be

* 공과대학 전자공학과 (Dept. of Electronics Engineering, Cheju Univ., Cheju-do, 690-756, Korea)

categorized into two classes at large. One approach which results in a suboptimal control law is based on the concept of optimal control sensitivity (Jamshidi and Zavarei, 1972). In this approach, the control law is expanded into a MacLaurin series in some parameters. The other one is to convert the TD problem to a nondelay problem (Zavarei, 1980). But these approaches are prohibitive to the LSTD systems such as a river pollution problem due to their computational burden.

To get around computationl difficulties which are associated with computational time and storage space, Tamura (1974) has proposed a multi-level method for LSTD systems by decomposition and coordination technique. The main disadvantage of Tamura's method is that it is necessary to perform linear search for the upper-level gradient algorithm. Hence the convergence rate is comparatively slow. Singh et al. (Singh, 1976; Singh *et. al*, 1981) have proposed a promising hierarchical algorithm by using interaction prediction method. This algorithm is found to be superior to other multi-level methods for a certain class of optimization problems. On the upper-level, it has more rapid convergence rate and fewer operations than other coordination rules such as linear search algorithm. But it also has a disadvantage that dimension of the given system has to be increased to transform the TD system into nondelay system.

In this paper, we describe an efficient hierarchical optimal control method for the LSTD systems based on the interaction prediction method without increasing the system dimension. The optimal tracking problem is transformed into a regulator problem with constant input by introducing a predetermined nominal input to the performance index. The steady-state tracking error for the method is determined analytically. Also, a necessary and sufficient condition for zero steady-state error is derived.

Problem Formulation

A schematic diagram of a river with multiple sewage work can be depicted as in Fig. 1 (Tamura, 1974).

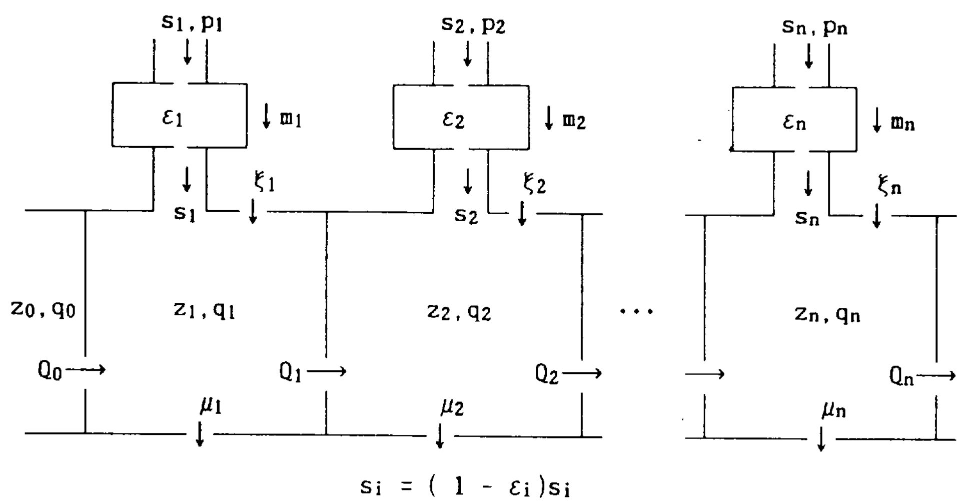


Fig. 1. A Schematic diagram of river system.

Then, from the mass balance considerations, we derive the following equations that govern the evolution in time of the BOD and DO concentrations.

$$\text{BOD}: z_i(k+1) - z_i(k) = -\alpha_i z_i(k) + \frac{Q_{i-1}(k)}{V_i} z_{i-1} - \frac{Q_i(k)}{V_i} z_i(k) + [1-\varepsilon_i(k)] \frac{s_i(k) m_i(k)}{V_i} \quad (1)$$

$$\text{DO}: q_i(k+1) - q_i(k) = -\alpha_i z_i(k) + \beta_i[q_i^s - q_i(k)] + \frac{Q_{i-1}(k)}{V_i} q_{i-1} - \frac{Q_i(k)}{V_i} q_i(k) - \mu_i(k) + \xi_i(k) + \frac{p_i(k) m_i(k)}{V_i} \quad (2)$$

$$(i = 1, 2 \cdots n, k = 1, 2 \cdots k_f - 1)$$

From above eqations, it is noted that the dynamics of DO is dependent on that of BOD. In these equations the symbols mean as followings.

$z_i(k)$: concentration of BOD in the ith reach at time k (mg/ℓ)

$q_i(k)$: concentration of DO in the ith reach at time k (mg/ℓ)

q_i^s : saturation concentration of DO in the ith reach (mg/ℓ)

$s_i(k)$: concentration of BOD in the effluent dischared to the ith reach at time k before treatment (mg/ℓ)

$m_i(k)$: volume of the effluent discharge in the reach during the time between k and k+1 (m³)

$\varepsilon_i(k)$: fraction of BOD removed from the effluent in the ith reach during the time between k and k+1

$Q_i(k)$: volume of water that flows from the ith reach to the (i+1) th reach during the time between k and k+1 (m³)

V_i: volume of water in the ith reach (m³)

$\mu_i(k)$: removal of DO from the ith reach by the effects of photosynthesis and respiration during the time between k and k+1 (mg/ℓ)

$\xi_i(k)$: addition of DO in the ith reach by the aeration (mg/ℓ) during the time between k and k+1

In (1) and (2) the terms z_{i-1} and q_{i-1} can be written as follows by taking into account the dispersion of BOD and DO concentrations.

$$z_{i-1} = \sum_{j=1}^{\theta_x} a_j z_{i-1}(k-j) \quad (3a)$$

$$q_{i-1} = \sum_{j=1}^{\theta_x} a_j q_{i-1}(k-j) \quad (3b)$$

$$\sum_{j=1}^{m} a_j = 1 \quad (3c)$$

The distributed delay model, (3) shows that for j = 1, 2, ⋯, m, fraction a_j of BOD and DO in the (i-1)th reach at time $(k-\theta_j)$ arrives in the ith reach at time k. This means that the dispersion delays are distributed in time between θ_1 and θ_m.

Let's define the state and the control vectors as :

$$x(k) = [z_1(k) q_1(k) z_2(k) q_2(k) \cdots z_n(k) \; q_n(k)]^T \quad (4a)$$

$$u(k) = [\varepsilon_1(k) \; \varepsilon_2(k) \cdots \varepsilon_n(k)]^T \quad (4b)$$

Then the following state space model can be obtained.

$$x(k+1) = A_0 x(k) + A_1 x(k-1) + \cdots + A_{\theta_x} x(k-\theta_x) + Bu(k) + c \quad (5a)$$

with initial conditions

$$x(k) = \phi_x(k), \; -\theta_x \le k \le 0 \quad (5b)$$

$$u(k) = \phi_u(k), \; -\theta_u \le k < 0 \quad (5c)$$

Without loss of generality, we assumed that the matrices A, B and c in (5a) are constant. In (5a), $A_i (i=0, 1, \cdots, \theta_x) \in R^{2n \times 2n}$ is a system matrix, $B \in R^{2n \times n}$ is an input matrix, $c \in R^{2n \times 1}$ is a constant input vector, θ_x is a delay in states. Let's define the performance index for the optimal tracking control problem as

$$J=\frac{1}{2}\sum_{k=0}^{kf-1}\{\|x(k)-x^d\|_Q^2+\|u(k)-u^n\|_R^2\} \qquad (6)$$

where $Q\in R^{2n\times 2n}$ is a state weighting matrix, $R\in R^{n\times n}$ is an input weighting matrix, $x^d\in R^{2n\times 1}$ is a constant desired or reference value of state vector and $u^n\in R^{n\times 1}$ is a predetermined nominal control input, which will be discussed more detail in the next section. It is assumed that Q and R are positive semi-definite and positive definite block diagonal matrix, respectively. Here, the optimization problem is to find a control law which causes the state vector of the system (5a) to follow a desired value that minimizes the performance index (6).

Define a new state and control vector as

$$z(k) \equiv x(k) - x^d \qquad (7a)$$

$$v(k) \equiv u(k) - u^n \qquad (7b)$$

Then the above optimal tracking problem can be transformed into a regulator problem with a constant input which is expressed as

$$z(k+1) = A_0 z(k) + A_1 z(k) + \cdots + A_{\theta_x} z(k-\theta_x) + Bv(k) + c^p \qquad (8a)$$

$$z(k) = \phi_x(k) - x^d, \ -\theta_x \le k \le 0 \qquad (8b)$$

$$v(k) = \phi_u(k) - u^n, \ -\theta_u \le k < 0 \qquad (8c)$$

$$J= \frac{1}{2}\sum_{k=0}^{kf-1}\{\|z(k)\|_Q^2+\|v(k)\|_R^2\} \qquad (9)$$

where

$$c^p = [\sum_{k=0}^{\theta x} A_k - I_n]x^d + Bu^n + c. \qquad (10)$$

It is prohibitive to use the centralized optimal control method to obtain the optimal solution for the above LSTD system due to computational burden. To overcome the computational difficulties associated with computational time and storage space, we develope a hierarchical technique based on the interaction prediction method.

Hierarchical Optimization

The above centralized optimal regulator problem for the LSTD system is decomposed into smaller subproblems to obtain the optimal solution in a hierarchical manner. The i-th subproblem is expressed as

$$z_i(k+1) = A_{ii}z_i(k) + B_{ii}v_i(k) + c_i^p + h_i(k) \qquad (11a)$$

$$h_i(k) = \sum_{(j\neq i,\ if\ l=0)}^{N} \{\sum_{l=0}^{\theta x} L_{ijl}\, z_j(k-l)\} + \sum_{j\neq i}^{N} M_{ij} v_j \qquad (11b)$$

$$z_i(k) = \phi_{xi}(k) - x_i^d, \ -\theta_x \le k \le 0 \qquad (11c)$$

$$v_i(k) = \phi_{ui}(k) - u_i^n, \ -\theta_u \le k < 0 \qquad (11d)$$

$$J_i = \frac{1}{2}\sum_{k=0}^{kf-1} \{\|z_i(k)\|_{Q_i}^2 + \|v_i(k)\|_{R_i}^2\} \qquad (12)$$

$$i=1, 2, \cdots, N$$

where $h_i(k) \in R^{n_i\times 1}$ consists of interaction inputs which come in from the other subsystems and time-delayed states of the i-th subsystem, $L_{ijl}\in R^{n_i\times n_j}$ is a coupling matrix of states, $M_{ij}\in R^{n_i\times m_j}$ is a coupling matrix of control inputs, N is the number of the interconnected subsystems which comprise the overall system, $\sum_{i=1}^{N} n_i=2n$ and $\sum_{i=1}^{N} m_i=n$.

Now, we use the interaction prediction method which is attractive due to simple upper-level algorithm and fast convergence rate. The interaction prediction method is essentially composed of obtaining optimal solutions of decomposed subproblems at lower-level and of updating the coordination vector to force the independent lower-level solutions to the optimal solution of the overall system.

First, consider the lower-level problem to find the optimal solutions for the decomposed subproblems. The Hamiltonian function for the i-th subsystem can be written as

$$H_i = \tfrac{1}{2} \| z_i(k) \|^2_{Q_i} + \tfrac{1}{2} \| v_i(k) \|^2_{R_i} + \gamma_i^T(k) h_i(k) - \sum_{(j\neq i,\, if\, l=0)}^{N} [\sum_{l=0}^{\theta_x} \gamma_i^T(k+1)\, L_{jil} z_i(k)] - \sum_{j\neq i}^{N} \gamma_j^T(k) M_{ji} v_i(k) + q_i^T(k+1)\, [A_{ii} z_i(k) + B_{ii} v_i(k) + c_i^p + h_i(k)\,] \qquad (13)$$

where $\gamma_i(k) \in R^{n_i x 1}$ and $q_i(k) \in R^{n_i x 1}$ are Lagrange multiplier and costate vector of i-th subsystem, respectively. From (13) the necessary conditions for optimality are obtained as

$$z_i(k+1) = A_{ii} z_i(k) + B_{ii} v_i(k) + c_i^p + h_i(k) \qquad (14a)$$

$$z_i(0) = \phi_{xi}(0) - x_i^d \qquad (14b)$$

$$v_i(k) = -R_i^{-1} [B_{ii}^T q_i(k+1) - \sum_{j\neq i}^{N} M_{ji}^T \gamma_j(k)\,] \qquad (14c)$$

$$\gamma_i(k) = 0, \quad (k \geq k_f) \qquad (14d)$$

$$q_i(k) = Q_i z_i(k) + A_{ii}^T q_i(k+1) - \sum_{(j\neq i,\, if\, l=0)}^{N} \sum_{l=0}^{\theta x} L_{jil}^T \gamma_j(k+1) \qquad (14e)$$

$$q_i(k_f) = 0 \qquad (14f)$$

Next, consider the upper-level problem in order to optimize the overall system by coordinating the lower-level solutions. For this purpose, the additively separable Lagrangian function can be written as

$$L = \sum_{i=1}^{N} \sum_{k=0}^{kf-1} \{ \tfrac{1}{2} \| z_i(k) \|^2_{Q_i} + \tfrac{1}{2} \| v_i(k) \|^2_{R_i} + \gamma_i^T(k) h_i(k) - \sum_{(j\neq i,\, if\, l=0)}^{N} \gamma_j^T(k) [\sum_{l=0}^{\theta x} L_{jil} z_i(k-1)] + \sum_{j\neq i}^{N} \gamma_j^T(k) M_{ji} v_i(k) + q_i^T(k+1) [A_{ii} z_i(k) + B_{ii} v_i(k) + c_i^p + h_i(k) - z_i(k+1)] \} \qquad (15)$$

Then the coordination rule at the upper-level from iteration L to L+1 is obtained by

$$\begin{bmatrix} \gamma_i(k) \\ h_i(k) \end{bmatrix}^{L+1} = \begin{bmatrix} -q_i(k+1) \\ \sum_{(j\neq i,\, if\, l=0)}^{N} [\sum_{l=0}^{\theta x} L_{ijl} z_j(k-1)] + \sum_{j\neq i}^{N} M_{ij} v_j(k) \end{bmatrix}^{L} \qquad (16)$$

Now, a step-by-step computational procedure to obtain optimal control law for the LSTD system is summarized.

step 1 : At the upper-level, set L=1 and predict initial values for $\gamma_i(k)$ and $h_i(k)$ (i=1, 2, ⋯, N, k=0, 1, ⋯, kf-1). Then pass them down to the lower-level.

step 2 : At the lower-level, solve the independent necessary conditions for optimality (14a)-(14f) by using $\gamma_i(k)$ and $h_i(k)$ passed from upper-level.

step 3 : At the upper-level, check the convergence of (16), i.e., whether their errors are within the predetermined error bounds, ε. If not, update $\gamma_i(k)$ and $h_i(k)$ from (16) by using $z_i(k)$, $v_i(k)$ and $q_i(k)$ passed from the lower-level. Then set L=L+1 and go to step 2.

step 4 : If step 3 is converged, calculate the optimal control law and state trajectory from (7a) and (7b), respectively.

Steady-State Considerations

If the final time kf is large enough for the system to reach a steady-state, the following Theorem can be applied.

Theorem 1 : If the proposed hierarchical algorithm in section 3 for the optimal control of the

LSTD system (5) and (6) converges and the inverse of $[In - \sum_{l=0}^{\theta_x} A_l]$ exists, the steady-state tracking error is given by

$$e_{ss} = -\{I_n - (\sum_{l=0}^{\theta_x} A_l) + B R^{-1} B^T [I_n - \sum_{l=0}^{\theta_x} A_l^T]^{-1} Q\}^{-1} c^p \quad (17)$$

Proof : If the algorithm converges we obtain the followings from (16)

$$\gamma_i(k) = -q_i(k+1) \quad (18a)$$

$$h_i(k) = \sum_{(j\neq i, \text{if } l=0)}^{N} [\sum_{l=0}^{\theta_x} L_{ijl} z_i(k-l)] + \sum_{j\neq i}^{N} M_{ij} v_i(k) \quad (18b)$$

Substituting (18a) and (18b) into the necessary conditions for optimality (14a)-(14f), we obtain the following integrated expressions :

$$z(k+1) = \sum_{l=0}^{\theta_x} A_l z(k-l) + B v(k) + c^p \quad (19)$$

$$v(k) = -R^{-1} B^T q(k+1) \quad (20)$$

$$q(k) = Qz(k) + \sum_{l=0}^{\theta_x} A_l^T q(k+l+1) \quad (21)$$

Since $z(k)$, $v(k)$ and $q(k)$ are constant vectors at steady-state, we have

$$z_s = \sum_{l=0}^{\theta_x} A_l z_s + B v_s + c^p \quad (22)$$

$$v_s = -R^{-1} B^T q_s \quad (23)$$

$$q_s = Qz_s + \sum_{l=0}^{\theta_x} A_l^T q_s \quad (24)$$

where the subscript s denotes steady-state. Substituting (23) and (24) into (22) we obtain

$$[I_n - \sum_{l=0}^{\theta_x} A_l] z_s = -[B R^{-1} B^T] [I_n - \sum_{l=0}^{\theta_x} A_l^T]^{-1} Q z_s + c^p \quad (25)$$

We define the steady-state tracking error as

$$e_{ss} = x^d - x_s \quad (26)$$

Then, taking into account (7a) and (26), we obtain (17) from (25). This completes the proof.

Remark 1 :

(a) It is noted that the quantity inside the braces on the righ-hand side of (17) is nonsingular if the inverse of $[I_n - \sum_{l=0}^{\theta_x} A_l]$ exists.

(b) Theorem I reveals that the steady-state tracking error can be obtained from the state equation and the performance index without solving the optimization problem.

(c) It is noted that an increase in $\|Q\|$ or a decrease in $\|R\|$ reduces the steady-state tracking error.

(d) (17) can be rearranged as

$$B R^{-1} B^T [I_n - \sum_{l=0}^{\theta_x} A_l^T]^{-1} Q e_{ss} = -c^p - [I_n - \sum_{l=0}^{\theta_x} A_l] e_{ss} \quad (27)$$

The above equation shows that when allowable steady-state error and the input weighiting matrix are given, the state weighting matrix can be determined if the right-hand side vector of (27) belongs to the column space of the left-hand side matrix of (27) except $Q e_{ss}$.

Remark 2 :

(a) From Theorem I and (10), the necessary and sufficient condition for zero steady-state tracking error is that a vector $[I_n - \sum_{l=0}^{\theta_x} A_l] x^d - c$ belongs to the column space of a matrix B.

(b) The steady-state tracking error does not exist regardless of Q and R if the necessary and sufficient condition for zero steady-state tracking error is satisfied. In this case, if B has full colum rank the nominal control input u^n is obtained by

$$u^n = [B^TB]^{-1}B^T \{[I_n - \sum_{l=0}^{\theta_x} A_l]x^d - c\} \quad (28)$$

(c) If the necessary and sufficient condition is not satisfied, the nominal control input obtained from (28) is a approximate least-square solution for $c^p=0$. In this case the steady-state tracking error is give as

$$e_{ss} = (I_n - (\sum_{l=0}^{\theta_x} A_l) + B\ R^{-1}B^T[I_n - \sum_{l=0}^{\theta_x} A_l^T]^{-1}Q)^{-1}$$

$$\{[I_n - B[B^TB]^{-1}B^T][I_n - \sum_{l=0}^{\theta_x} A_l]^{-1}x^d - c]\} \quad (29)$$

if B has full column rank.

Numerical Example

To illustrate the algorithm, river pollution model of River Cam outside Cambridge, England (Tamura, 1974) is considered. The numerical values for the model are $N=2$, $n_i=2$, $m_i=1\,(i=1, 2)$, $\theta_x=2$,

$$A_{ii} = \begin{bmatrix} 0.18 & 0. \\ -0.25 & 0.27 \end{bmatrix}, \quad B_{ii} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \ (i=1,2)$$

$$c_1 = [4.5\ 6.15]^T, \quad c_2 = [2\ 2.65]^T,$$

$$L_{121} = L_{111} = L_{121} = L_{221} = L_{112} = L_{122} = L_{222} = 0,$$

$$L_{210} = \begin{bmatrix} 0.0825 & 0. \\ 0. & 0.0825 \end{bmatrix},$$

$$L_{211} = \begin{bmatrix} 0.0825 & 0. \\ 0. & 0.0385 \end{bmatrix},$$

$$L_{212} = \begin{bmatrix} 0.0825 & 0. \\ 0. & 0.0825 \end{bmatrix}.$$

We have chosen that $Q_i=I_2$, $R_i=100$, $\varepsilon=10^{-5}$ and $kf=30$ which is large enough for the system to reach steady-state. Simulations are carried out for the following two cases.

Case 1 : The necessary and sufficient condition for zero steady-state tracking error is satisfied : $x_1^d = [4.16\ 7.0]^T$ and $x_2^d = [5.56\ 7.0]^T$.

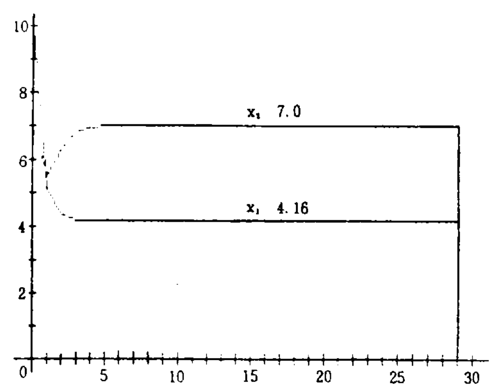
Case 2 : The necessary and sufficient condition for zero steady-state tracking is not satisfied; $x_1^d = [5.0\ 7.0]^T$ and $x_2^d = [5.0\ 7.0]^T$

The simulation results for the Tamura's method and proposed method are summarized in Table 1.

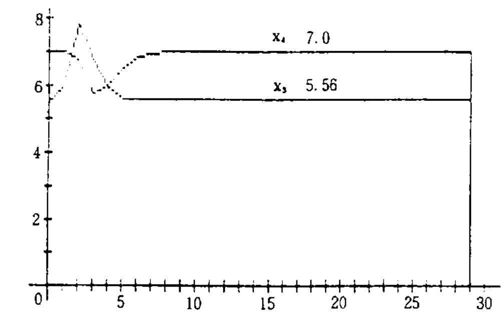
Table 1. Summary of the simulation results

method	iteration number	steady-state tracking error	
		case 1	case 2
Tamura's method	16	[-1.48 .01 -1.36 .78]^T	[-.61 .16 -1.20 .34]^T
proposed method	9	0.	[0. .29 0. .02]^T

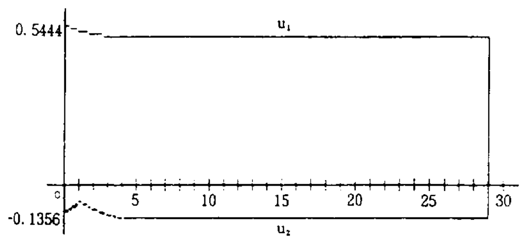
It is important to note that the proposed method is advantageous over the other methods in steady-state tracking error and convergence rate. The steady-state tracking error resulted from the proposed hierarchical algorithm is consistent with Theorem 1. Note that the steady-state tracking error of the proposed method in case 1 is zero. Also, the optimal trajectories of state variables and control inputs for the case 1 are shown in Fig. 2.



(a) State variables of subsystem 1



(b) State variables of subsystem 2



(c) Control inputs

Fig. 2. Optimal trajectories of state variables and control inputs.

Conclusion

A large-scale discrete-time state space model for a multiple-reach river system in obtained by putting BOD and DO concentrations as state variables. A hierarchical optimal control algorithm which is applicable to the river pollution model is developed using the interaction prediction method. The steady-state tracking error is determined analytically and a necessary and sufficient condition for zero steady-state error is derived.

Computer simulation for the river pollution model reveals that the proposed method has better steady-state response and fewer upper-level interations than Tamura's method.

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〈국문초록〉

강의 수질오염 제어를 위한 계층적 최적제어

생화학적 산소요구량 및 용해된 산소량을 이용하여 여러구간이 있는 강에 대한 이산 상태공간모델을 기술하였다. 상호작용 예측방법을 이용하여, 상태변수에 시간지연이 존재하는 대규모 시스템에 적용가능한 계층적 최적화 방법을 기술하였다. 정상상태 오차를 해석적으로 구하고, 상수 목표치 추적문제에 있어서 정상상태 오차가 발생하지 않을 필요충분조건을 규명하였다. 수질오염 모델에 대한 컴퓨터 모사를 통하여 기술한 알고리듬의 타당성을 확인하였다.