

## SEMINORMAL OPERATORS AND WEYL SPECTRA

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**ABSTRACT.** In this paper we show that the Weyl spectrum of a seminormal operator  $T$  satisfies the spectral mapping theorem for any analytic function  $f$  on a neighborhood of  $\sigma(T)$  and Weyl's theorem holds for  $f(T)$ . Finally we give conditions for an operator to be of the form unitary + compact and answer an old question of Oberai.

**0. Introduction.** Throughout this paper let  $H$  denote an infinite dimensional Hilbert space and  $B(H)$  the set of all bounded linear operators on  $H$ . If  $T \in B(H)$ , we write  $\sigma(T)$  for the spectrum of  $T$ ,  $\pi_0(T)$  for the set of eigenvalues of  $T$ ,  $\pi_{0f}(T)$  for the set of eigenvalues of finite multiplicity, and  $\pi_{00}(T)$  for the isolated points of  $\sigma(T)$  that are eigenvalues of finite multiplicity. If  $E$  is a subset of  $\mathbb{C}$ , we write  $\text{iso } E$  for the set of isolated points of  $E$ . An operator  $T \in B(H)$  is said to be *Fredholm* if its range  $\text{ran } T$  is closed and both the null spaces  $\ker T$  and  $\ker T^*$  are finite dimensional. The *index* of a Fredholm operator  $T$ , denoted by  $i(T)$ , is defined by

$$i(T) = \dim \ker T - \dim \ker T^*.$$

The *essential spectrum* of  $T$ , denoted by  $\sigma_e(T)$ , is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl operator*. The *Weyl spectrum* of  $T$ , denoted by  $\omega(T)$ , is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$

It was shown ([2]) that for any operator  $T$ ,  $\sigma_e(T) \subset \omega(T) \subset \sigma(T)$ , and  $\omega(T)$  is a nonempty compact subset of  $\mathbb{C}$ .

Recall ([9], [12]) that an operator  $T \in B(H)$  is said to be *seminormal* if either  $T$  or  $T^*$  is hyponormal. Every hyponormal operator is seminormal,

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but the converse is not true in general. Unilateral shifts are examples of seminormal operators.

An operator  $T \in B(H)$  is said to be *dominant* ([4], [13]) if for every  $z \in \mathbb{C}$  there exists  $M_z > 0$  such that

$$(T - z)(T - z)^* \leq M_z(T - z)^*(T - z)$$

In this case, if  $\sup_{z \in \mathbb{C}} M_z = M < \infty$ ,  $T$  is said to be  $M$ -hyponormal, and if  $M = 1$ ,  $T$  is hyponormal. Evidently,

$$T \text{ is hyponormal} \implies T \text{ is } M\text{-hyponormal} \implies T \text{ is dominant}$$

If  $T$  is both Fredholm and seminormal, then either  $i(T) \leq 0$  or  $i(T) \geq 0$ . It was known that the mapping  $T \rightarrow \omega(T)$  is upper semi-continuous, but not continuous at  $T$  ([7]). However if  $T_n \rightarrow T$  with  $T_n T = T T_n$  for all  $n \in \mathbb{N}$  then

$$(0.1) \quad \lim \omega(T_n) = \omega(T).$$

It was known that  $\omega(T)$  satisfies the one-way spectral mapping theorem for analytic functions: if  $f$  is analytic on a neighborhood of  $\sigma(T)$  then

$$(0.2) \quad \omega(f(T)) \subset f(\omega(T)).$$

The inclusion (0.2) may be proper (see [2, Example 3.3]). If  $T$  is normal then  $\sigma_e(T)$  and  $\omega(T)$  coincide. Thus if  $T$  is normal and  $f$  is analytic on a neighborhood of  $\sigma(T)$ , it follows that  $\omega(f(T)) = f(\omega(T))$  since  $f(T)$  is also normal.

In this paper we show that the Weyl spectrum of a seminormal operator  $T$  satisfies the spectral mapping theorem for any analytic function  $f$  on a neighborhood of  $\sigma(T)$  and Weyl's theorem holds for  $f(T)$ . Finally we give conditions for an operator to be of the form unitary + compact and answer an old question of Oberai.

**1. Weyl spectral properties.** It was shown ([2]) that for any operator  $T$ ,  $\omega(T^*) = \omega(T)^*$  and

$$\omega(T) = \bigcap_{K \in \mathcal{K}} \sigma(T + K).$$

The Weyl spectrum of an operator is the disjoint union of the essential spectrum and a particular open set ([2]): For any operator  $T$  in  $B(H)$ ,

$$(1.1) \quad \omega(T) = \sigma_e(T) \cup \{\lambda : T - \lambda \text{ is Fredholm and } i(T - \lambda) \neq 0\}.$$

We have the concrete form of  $\omega(T)$  provided  $T$  is seminormal:

THEOREM 1. *If  $T$  is a seminormal operator, then*

$$\omega(T) = \cap \{ \sigma(T + K) : TK = KT \text{ and } K \text{ is normal compact} \}.$$

*Proof.* Let  $E = \{ \sigma(T + K) : TK = KT \text{ and } K \text{ is normal compact} \}$ . Then by [2, Theorem 2.5],  $\omega(T) \subset E$ . Since  $T$  is seminormal, by [10] there exists a normal compact operator  $K$  such that  $KT = TK$  and  $\sigma(T + K) = \omega(T)$ . Thus  $E \subset \omega(T)$ . This completes the proof.

THEOREM 2. *If  $\pi(T)$  is seminormal in  $B(H)/\mathcal{K}$  and if  $\omega(T) \subset \{ \lambda : |\lambda| = 1 \}$ , then  $T$  is of form unitary + compact.*

*Proof.* By hypothesis, 0 is not in  $\omega(T)$  and so  $T = S + K$ , where  $S$  is invertible and  $K$  is compact. Hence  $\pi(T) = \pi(S)$ . Since  $\sigma(\pi(T)) \subset \omega(T) \subset \{ \lambda : |\lambda| = 1 \}$ , by [9, p. 59 Corollary],  $\pi(T)$  is unitary in  $B(H)/\mathcal{K}$  and so  $\pi(S^*S) = \pi(I)$ . But square roots of a positive element of a  $C^*$ -algebra are unique, so  $\pi((S^*S)^{1/2}) = \pi(I)$ . Let the polar decomposition of  $S$  be given by  $S = U(S^*S)^{1/2}$ , where  $U$  is unitary. Then

$$\begin{aligned} \pi(T) &= \pi(S) = \pi(U(S^*S)^{1/2}) = \pi(U)\pi((S^*S)^{1/2}) \\ &= \pi(U)\pi(I) = \pi(U), \end{aligned}$$

so that  $T - U$  is compact.

COROLLARY 3. *If  $\pi(T)$  is normal in  $B(H)/\mathcal{K}$  and if  $\omega(T) \subset \{ \lambda : |\lambda| = 1 \}$ , then  $T$  is of form unitary + compact.*

For an example, consider  $T = U \oplus U^*$ , where  $U$  is the unilateral shift. In this case,  $\omega(T) = \{ \lambda : |\lambda| = 1 \} = \sigma_e(T)$ . But  $T$  is not a normal operator. Since  $I - UU^*$  and  $UU^* - I$  are rank one operators,  $\pi(T)$  is normal. By Corollary 3,  $T = U \oplus U^*$  is of the form unitary + compact.

LEMMA 4([13]). *If  $S$  and  $T$  are commuting dominant operators, then*

$$(1.2) \quad S, T \text{ Weyl} \iff ST \text{ Weyl}.$$

If the “dominant” condition is dropped in the above lemma, then the backward implication may fail even though  $T_1$  and  $T_2$  commute: For example, if  $U$  is the unilateral shift on  $l_2$ , consider the following operators on  $l_2 \oplus l_2$ :  $T_1 = U \oplus I$  and  $T_2 = I \oplus U^*$ .

THEOREM 5. If  $T$  is seminormal and  $f$  is analytic on a neighborhood of  $\sigma(T)$ , then  $\omega(f(T)) = f(\omega(T))$ .

*Proof.* If  $T$  is hyponormal, then it follows from [13, Theorem 2.2]. Suppose that  $T^*$  is hyponormal and  $p(t)$  is any polynomial. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I).$$

Since  $T^*$  is hyponormal,  $T^* - \bar{\mu}_i I$  are commuting hyponormal operators for each  $i = 1, 2, \dots, n$ . It thus follows from Lemma 4 and  $\omega(T^*) = \omega(T)^*$  that

$$\begin{aligned} \lambda \notin \omega(p(T)) &\iff p(T) - \lambda I = \text{Weyl} \\ &\iff a_0(T - \mu_1 I) \cdots (T - \mu_n I) = \text{Weyl} \\ &\iff \bar{a}_0(T^* - \bar{\mu}_1 I) \cdots (T^* - \bar{\mu}_n I) = \text{Weyl} \\ &\iff T^* - \bar{\mu}_i I = \text{Weyl for each } i = 1, 2, \dots, n \\ &\iff T - \mu_i I = \text{Weyl for each } i = 1, 2, \dots, n \\ &\iff \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \dots, n \\ &\iff \lambda \notin p(\omega(T)) \end{aligned}$$

which says that

$$(1.3) \quad \omega(p(T)) = p(\omega(T)).$$

Next suppose  $r$  is any rational function with no poles in  $\sigma(T)$ . Write  $r = p/q$ , where  $p$  and  $q$  are polynomials and  $q$  has no zeros in  $\sigma(T)$ . Then

$$r(T) - \lambda I = (p - \lambda q)(T)(q(T))^{-1}.$$

By (1.3),

$$(p - \lambda q)(T) \text{ Weyl} \iff p - \lambda q \text{ has no zeros in } \omega(T).$$

Thus we have

$$\begin{aligned} \lambda \notin \omega(r(T)) &\iff (p - \lambda q)(T) = \text{Weyl} \\ &\iff p - \lambda q \text{ has no zeros in } \omega(T) \\ &\iff ((p - \lambda q)(x))q(x)^{-1} \neq 0 \text{ for any } x \in \omega(T) \\ &\iff \lambda \notin r(\omega(T)) \end{aligned}$$

which says that  $\omega(r(T)) = r(\omega(T))$ . If  $f$  is analytic on a neighborhood of  $\sigma(T)$ , then by Runge's theorem ([4]), there is a sequence  $\{r_n(t)\}$  of rational functions with no poles in  $\sigma(T)$  such that  $\{r_n\}$  converges to  $f$  uniformly on a neighborhood of  $\sigma(T)$ . Since  $\{r_n(T)\}$  converges to  $f(T)$  and each  $r_n(T)$  commutes with  $f(T)$ , by [7]

$$f(\omega(T)) = \lim r_n(\omega(T)) = \lim \omega(r_n(T)) = \omega(f(T)).$$

An operator  $T$  is said to be *polynomially compact* ([2]) if there exists a polynomial  $p$  such that  $p(T)$  is compact. Thus we see that  $T$  is polynomially compact if and only if  $T^*$  is polynomially compact. From Theorem 5 and [2, Corollary 6.6], we can obtain the following result:

**THEOREM 6.** *If  $T$  is seminormal and satisfies condition (i), then  $T$  is normal ( $i = 1, 2, 3$ ).*

- (1)  $T$  is polynomially compact.
- (2) There exists an analytic function  $f$  on  $\sigma(T)$  such that  $f(T)$  is compact and  $f$  has finitely many zeros on  $\omega(T)$ .
- (3)  $\omega(T)$  is finite.

We say that *Weyl's theorem holds for  $T$*  if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including normal, hyponormal, and seminormal operators on a Hilbert space for which Weyl's theorem holds. Also it was shown in [8] that Weyl's theorem holds for any spectral operator of finite type on a Banach space. Oberai has raised the following question: Does there exist a hyponormal operator  $T$  such that Weyl's theorem does not hold for  $T^2$ ? Note that  $T^2$  may not be hyponormal even if  $T$  is hyponormal ([5, Problem 209]). We will show that Weyl's theorem holds for  $p(T)$  when  $T$  is a seminormal operator and  $p$  is a polynomial. Thus we answer the old question of Oberai since every hyponormal operator is seminormal.

Recall ([8]) that  $T \in B(H)$  is said to be *isoloid* if  $\text{iso } \sigma(T) \subset \pi_0(T)$ .

**LEMMA 7**([8]). *If  $T \in B(H)$  is isoloid and  $f$  is analytic on a neighborhood of  $\sigma(T)$ , then  $f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T))$ .*

**THEOREM 8.** *If  $T \in B(H)$  is seminormal, then for any analytic function  $f$  on a neighborhood of  $\sigma(T)$ , Weyl's theorem holds for  $f(T)$ .*

*Proof.* By [7] and [12], every seminormal operator  $T$  is isoloid and Weyl's theorem holds for any seminormal operator  $T$ . Hence by Theorem 5 and Lemma 7,

$$\omega(f(T)) = f(\omega(T)) = f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)).$$

Therefore Weyl's theorem holds for  $f(T)$ .

Since every hyponormal operator is seminormal, we obtain the following result which answers to the old question of Oberai.

**COROLLARY 9.** *If  $T \in B(H)$  is hyponormal, then for any polynomial  $p(t)$  Weyl's theorem holds for  $p(T)$ .*

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