

JOINT SPATIAL NUMERICAL RANGES OF OPERATORS ON BANACH SPACES

YOUNG OH YANG

1. Introduction

Throughout this paper, X will always denote a Banach space over the complex numbers \mathbb{C} , and $L(X)$ will denote the Banach algebra of all continuous linear operators on X . Operator will always mean continuous linear operator. An n -tuple of operators T_1, \dots, T_n on X will be denoted by $\hat{T} = (T_1, \dots, T_n)$. Let $L^n(X)$ be the set of all n -tuples of operators on X . X' will denote the dual space of X , $S(X)$ its unit sphere and $\Pi(X)$ the subset of $X \times X'$ defined by

$$\Pi(X) = \{(x, f) \in X \times X' : \|x\| = \|f\| = f(x) = 1\}.$$

For each $T \in L(X)$, we let T^* denote the dual operator. Given a subset B of X , we let \bar{B} denote the closure of B . If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we let $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$.

Let T be an operator on X . It is well known that the spatial numerical range $V(T)$ of T is not closed, not in general convex, but that it is always connected [1]. Also $\overline{V(\cdot)}$ is a continuous mapping of $L(X)$, endowed with norm topology to the set of compact subsets of \mathbb{C} , endowed with the Hausdorff metric topology. In this paper, we will define joint lower numerical range $LV(\hat{T})$ of $\hat{T} \in L^n(X)$, and study the relation between this concept and joint spatial numerical range $V(\hat{T})$, continuous properties of several mappings related to joint numerical ranges and analogous topological properties (connectedness, closedness etc) for an n -tuple of operators. Finally we will give some problems about joint spatial numerical range.

2. Joint spatial numerical ranges

Let $\hat{T} = (T_1, \dots, T_n) \in L^n(X)$ be an n -tuple of operators on X . The

joint spatial numerical range, joint operator norm and joint spatial numerical radius of \hat{T} respectively are defined by

$$V(\hat{T}) = \{(f(T_1x), \dots, f(T_nx)) \in \mathbf{C}^n : (x, f) \in \Pi(X)\},$$

$$\|\hat{T}\| = \sup \{(\sum_{i=1}^n \|T_i x\|^2)^{1/2} : \|x\| = 1, x \in X\},$$

and

$$\nu(\hat{T}) = \sup \{(\sum_{i=1}^n |z_i|^2)^{1/2} : (z_1, \dots, z_n) \in V(\hat{T})\}$$

respectively ([3], [7]). We also define another joint operator norm of T by $|\hat{T}| = (\sum_{i=1}^n \|T_i\|^2)^{1/2}$. Clearly $\nu(\hat{T}) \leq \|\hat{T}\| \leq |\hat{T}|$.

Given $x \in S(X)$, let $D(X, x) = \{f \in S(X') : f(x) = 1\}$, and $V(\hat{T}, x) = \{(f(T_1x), \dots, f(T_nx)) : f \in D(X, x)\}$. Clearly by the Hahn-Banach theorem, $V(\hat{T}, x) \neq \phi$, $V(\hat{T}) = \cup \{V(\hat{T}, x) : x \in S(X)\}$, and $V(\hat{T})$ is a nonempty and bounded subset of \mathbf{C}^n .

Given a_1, \dots, a_n of a unital normed algebra A , let $V(A; a_1, \dots, a_n) = \{(f(a_1), \dots, f(a_n)) : f \in A', f(1) = \|f\| = 1\}$ denote the joint numerical range of a_1, \dots, a_n [1]. Clearly $V(A; a_1, \dots, a_n)$ is a compact convex subset of \mathbf{C}^n . Also for each n -tuple $\hat{T} = (T_1, \dots, T_n)$ of operators on X ,

$$V(\hat{T}) \subset V(B(X); \hat{T}) = V(B(X); T_1, \dots, T_n).$$

LEMMA 2.1. Let $x \in S(X)$ and $\hat{T} = (T_1, \dots, T_n) \in L^n(X)$ be an n -tuple of operators on X . Then $V(\hat{T}, x)$ is the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$ such that for each $i = 1, \dots, n$,

$$|\lambda_i - \mu_i| < \|(T_i - \mu_i I)x\| \quad (\mu = (\mu_1, \dots, \mu_n) \in \mathbf{C}^n).$$

Proof. The proof is similar to that of Lemma 15.1 [2].

Let $X_{\mathbf{R}}$ denote the space X regarded as a Banach space over \mathbf{R} . Then the mapping $f \rightarrow \text{Re } f$ is an isometric real linear mapping of X' onto $X'_{\mathbf{R}}$, and the mapping $(x, f) \rightarrow (x, \text{Re } f)$ maps $\Pi(X)$ onto $\Pi(X_{\mathbf{R}})$ [2]. From these facts, we obtain the following result.

THEOREM 2.2. Let $\hat{T} = (T_1, \dots, T_n) \in L^n(X)$, and $\hat{T}_{\mathbf{R}}$ denote \hat{T} regarded as an n -tuple of operators on $X_{\mathbf{R}}$. Then

$$V(\hat{T}_{\mathbf{R}}) = \text{Re } V(\hat{T}) = \{(\text{Re } f(T_1x), \dots, \text{Re } f(T_nx)) : (x, f) \in \Pi(X)\}.$$

LEMMA 2.3 ([2]). Let X, Y be metric spaces with Y compact, and let ϕ be a mapping of X into 2^Y such that $\phi(x)$ is closed for each $x \in X$. Then ϕ is upper semi-continuous (u. s. c) if and only if

$$x_n \in X, y_n \in \phi(x_n) \quad (n = 1, 2, \dots), \quad x = \lim_{n \rightarrow \infty} x_n, y = \lim_{n \rightarrow \infty} y_n \quad \text{imply } y \in \phi(x).$$

From this lemma and the similar method of proof in Lemma 15.8 [2], we obtain the following result.

THEOREM 2.4. *For each $\hat{T} \in L^n(X)$, the mapping $x \rightarrow V(\hat{T}, x)$ is an upper semi-continuous mapping of $S(X)$ with the norm topology into the nonvoid compact convex subsets of C^n .*

$L^n(X)$ becomes an algebra with involution if we define all the operations componentswise. In particular, if $\hat{S} = (S_1, \dots, S_n)$ and $\hat{T} = (T_1, \dots, T_n)$ are elements of $L^n(X)$, we have $\hat{S}^* = (S_1^*, \dots, S_n^*)$, $\hat{S}\hat{T} = (S_1T_1, \dots, S_nT_n)$, and a norm is defined by $\|\hat{T}\| = \sup \{ (\sum_{i=1}^n \|T_i x\|^2)^{1/2} : x \in X, \|x\| = 1 \}$.

THEOREM 2.5. *The mapping $\hat{T} \rightarrow \overline{V(\hat{T})}$ is a continuous mapping from $L^n(X)$, endowed with a norm topology to the set of compact subsets of C^n , endowed with the Hausdorff metric topology. Also $\nu(\cdot)$ is a continuous real-valued mapping on $L^n(X)$.*

Proof. Let $\hat{S} = (S_1, \dots, S_n)$ and $\hat{T} = (T_1, \dots, T_n)$ be any elements of $L^n(X)$. If $\|\hat{S} - \hat{T}\| < \varepsilon$ and $(x, f) \in \Pi(X)$, then

$$\begin{aligned} & |(f(S_1x), \dots, f(S_nx)) - (f(T_1x), \dots, f(T_nx))| \\ &= |(f((S_1 - T_1)x), \dots, f((S_n - T_n)x))| \\ &= (\sum_{i=1}^n |f((S_i - T_i)x)|^2)^{1/2} \leq \|\hat{S} - \hat{T}\|, \end{aligned}$$

and so

$$\begin{aligned} (f(S_1x), \dots, f(S_nx)) &= (f(T_1x), \dots, f(T_nx)) + \\ &\quad (f((S_1 - T_1)x), \dots, f((S_n - T_n)x)) \\ &\in V(\hat{T}) + \varepsilon. \end{aligned}$$

Thus $\overline{V(\hat{S})} \subset \overline{V(\hat{T})} + (\varepsilon)$. By symmetry, $\overline{V(\hat{T})} \subset \overline{V(\hat{S})} + (\varepsilon)$. Thus $\|\hat{S} - \hat{T}\| < \varepsilon$ implies $d(\overline{V(\hat{S})}, \overline{V(\hat{T})}) \leq \varepsilon$, and $\overline{V(\cdot)}$ is a continuous mapping from $L^n(X)$ to the set of compact subsets of C^n , endowed with the Hausdorff metric topology.

Also $\nu(\hat{S}) \leq \nu(\hat{T}) + \varepsilon$ and $\nu(\hat{T}) \leq \nu(\hat{S}) + \varepsilon$ imply $|\nu(\hat{S}) - \nu(\hat{T})| \leq \varepsilon$. So $\nu(\cdot)$ is a continuous real-valued mapping on $L^n(X)$.

Given an n -tuple $\hat{T} = (T_1, \dots, T_n)$ of operators on X' , the joint lower numerical range $LV(\hat{T})$ of \hat{T} is defined by

$$LV(\hat{T}) = \{ ((T_1f)x, \dots, (T_nf)x) : (x, f) \in \Pi(X) \}.$$

THEOREM 2.6. Let $\hat{T} = (T_1, \dots, T_n)$ be an n -tuple of operators on X' (not necessarily commuting!). Then

$$LV(\hat{T}) \subset V(\hat{T}) \subset \overline{LV(\hat{T})}.$$

Proof. Let \hat{x} denote the canonical image in X'' of $x \in X$. Then $(x, f) \in \Pi(X)$ implies $(f, \hat{x}) \in \Pi(X')$, and so $(T_j f)(x) = \hat{x}(T_j f)$ ($j=1, \dots, n$). Hence $LV(\hat{T}) \subset V(\hat{T})$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in V(\hat{T})$ i.e., for each $j=1, \dots, n$, $\lambda_j = \phi(T_j f)$ with $(f, \phi) \in \Pi(X')$. Let ε be an arbitrary number with $0 < \varepsilon < 1$, and let X_1 denote the closed unit ball of X . Since \hat{X}_1 is weak*-dense in X_1'' , there exist $x \in X_1$ with

$|\phi(f) - \hat{x}(f)| < (\varepsilon/2)^2$, $|\phi(T_j f) - \hat{x}(T_j f)| < \varepsilon/\sqrt{n}$ ($j=1, \dots, n$). We have $x \in X_1$, $f \in S(X')$, and $|1 - f(x)| < (\varepsilon/2)^2$. Therefore by the Bishop-Phelps-Bollobás theorem [2], there exists $(y, g) \in \Pi(X)$ such that $\|y - x\| < \varepsilon$, $\|f - g\| < \varepsilon$. Then $(\hat{T}g)(y) = ((T_1 g)y, \dots, (T_n g)y) \in LV(\hat{T})$ and

$$\begin{aligned} |\lambda - (\hat{T}g)(y)| &\leq |\lambda - \hat{x}(\hat{T}f)| + |(\hat{T}f)(x) - (\hat{T}f)(y)| \\ &\quad + |(\hat{T}f)(y) - (\hat{T}g)(y)| \\ &< \varepsilon + |\hat{T}| \|x - y\| + |\hat{T}| \|f - g\| \\ &< \varepsilon(1 + 2|\hat{T}|). \end{aligned}$$

Since ε is arbitrary, $\lambda \in \overline{LV(\hat{T})}$.

COROLLARY 2.6. For each $\hat{T} = (T_1, \dots, T_n) \in L^n(X)$, $V(\hat{T}) \subset V(\hat{T}^*) \subset \overline{V(\hat{T})}$.

Proof. This follows from the above theorem and the fact that $LV(\hat{T}^*) = V(\hat{T})$.

The following example shows that we can have $V(\hat{T}) \neq V(\hat{T}^*)$.

EXAMPLE 2.7. Let $X = c_0$ and define $T \in B(X)$ by

$(Tx)(n) = \sum_{k=0}^{\infty} 2^{-k-1} x(n+k)$ ($n=1, 2, 3, \dots$). Then $V(T, T) \neq V(T^*, T^*)$ i.e., $V(\hat{T}) \neq V(\hat{T}^*)$ for $\hat{T} = (T, T)$.

Proof. It is clear that $\|Tx\| < 1$ for $\|x\| = 1$, and so $(1, 1) \notin V(\hat{T})$. With the normal identifications, $X' = l^1$, $X'' = l^\infty$, and $(T^*f)(n) = \sum_{k=1}^n 2^{k-n-1} f(k)$ ($n=1, 2, \dots$). Let $f \in X'$, $\phi \in X''$ be defined by $f(n) = \delta_n^1$, $\phi(n) = 1$ ($n=1, 2, \dots$). Then $(f, \phi) \in \Pi(X')$ and $(1, 1) = (\phi(T^*f), \phi(T^*f)) \in V(\hat{T}^*)$.

Let $(X, \|\cdot\|)$ denote a complex Banach space and let $N(X)$ denote the set of all norms on X equivalent to $\|\cdot\|$. Given $p, q \in N(X)$, we define $\mu(p, q)$, $d(p, q)$ by

$$\begin{aligned} \mu(p, q) &= \inf \{ \delta \geq 1 : 1/\delta \leq p(x)/q(x) \leq \delta, x \in X - \{0\} \}, \\ d(p, q) &= \log \mu(p, q). \end{aligned}$$

We note that d is a metric on $N(X)$, and $(N(X), d)$ is complete [2].

From now on, it is convenient to denote also by p the dual norm of p . The set Π_p is then defined by

$$\Pi_p = \{ (x, f) \in X \times X' : p(x) = p(f) = f(x) = 1 \}, \text{ and given } \hat{T} = (T_1, \dots, T_n) \in L^n(X), V_p(\hat{T}) = \{ (f(T_1x), \dots, f(T_nx)) : (x, f) \in \Pi_p \}.$$

THEOREM 2.8. *The mapping $p \longrightarrow \overline{V_p(\hat{T})}$ is a continuous mapping from $N(X)$, endowed with the above defined metric d to the set of compact subsets of \mathbb{C}^n , endowed with the Hausdorff metric topology.*

Proof. With $\kappa > 1$, we let $G_\kappa = \{ p \in N(X) : \mu(p, \|\cdot\|) < \kappa \}$. Then by Theorem 18.3 [2], $p \longrightarrow \Pi_p$ is uniformly continuous on G_κ . Given $\varepsilon > 0$, choose $\delta > 0$ such that $p, q \in G_\kappa$, $d(p, q) < \delta$ implies $d(\Pi_p, \Pi_q) < \varepsilon$. Given $\lambda = (\lambda_1, \dots, \lambda_n) \in V_q(\hat{T})$, we have $\lambda = (f(T_1x), \dots, f(T_nx))$ for some $(x, f) \in \Pi_q$. Then $d((x, f), \Pi_p) < \varepsilon$ and so there exists $(y, g) \in \Pi_p$ with $\|x - y\| + \|f - g\| < \varepsilon$. Then

$$\begin{aligned} & | (g(T_1y), \dots, g(T_ny)) - (f(T_1x), \dots, f(T_nx)) | \\ & \leq | (g(T_1y), \dots, g(T_ny)) - (f(T_1y), \dots, f(T_ny)) | \\ & \quad + | (f(T_1y), \dots, f(T_ny)) - (f(T_1x), \dots, f(T_nx)) | \\ & \leq \|g - f\| \|\hat{T}\| \|y\| + \|f\| \|\hat{T}\| \|y - x\| \\ & \leq \|g - f\| \|\hat{T}\| \kappa + \kappa \|\hat{T}\| \|y - x\| < \varepsilon \kappa \|\hat{T}\|. \end{aligned}$$

Thus $\sup \{ d(\lambda, \overline{V_p(\hat{T})}) : \lambda \in V_q(\hat{T}) \} \leq \varepsilon \kappa \|\hat{T}\|$, and by symmetry,

$\sup \{ d(\lambda, \overline{V_q(\hat{T})}) : \lambda \in V_p(\hat{T}) \} \leq \varepsilon \kappa \|\hat{T}\|$. Hence

$$d(\overline{V_q(\hat{T})}, \overline{V_p(\hat{T})}) \leq \varepsilon \kappa \|\hat{T}\|.$$

3. Topological properties of joint numerical ranges

For every $T \in L(X)$, it is well-known that the spatial numerical range $V(T)$ of T is not in general convex, not closed, but that it is always connected. The norm \times weak* topology is defined as the product topology on $X \times X'$ given by the norm topology on X and the weak* topology on X' [1].

LEMMA 3.1. *Let π_1 denote the natural projection $\pi_1(x, f) = x$ of $X \times X'$*

onto X , and for each $p \in N(X)$, let E be a subset of Π_p that is relatively closed in Π_p with respect to the norm \times weak* topology. Then $\pi_1(E)$ is a norm closed subset of X .

THEOREM 3.2. For each $p \in N(X)$, Π_p is a connected subset of $X \times X'$ with the norm \times weak* topology unless X has dimension one over \mathbf{R} .

Proof. The proof is similar to that of Theorem 11.4 [1].

COROLLARY 3.3. For each $p \in N(X)$, $V_p(\hat{T})$ is connected.

Proof. We have

$$|f(T_i x) - g(T_i y)| \leq p(T_i x - T_i y) + |(f - g)(T_i y)|$$

$((x, f), (y, g) \in \Pi_p, i = 1, 2, \dots, n)$. Therefore the mapping $(x, f) \rightarrow f(T_i x)$ is a continuous mapping of Π_p with the relative norm \times weak* topology onto $V_p(T_i)$. Since the natural projection $\pi_i: \mathbf{C}^n \rightarrow \mathbf{C}$ is continuous, the mapping $(x, f) \rightarrow (f(T_1 x), \dots, f(T_n x))$ is a continuous mapping of Π_p with the relative norm \times weak* topology onto $V(\hat{T})$. By the above theorem, $V(\hat{T})$ is connected, except perhaps when X has dimension one over \mathbf{R} .

Finally let X have dimension one over \mathbf{R} , and take $u \in X$ with $p(u) = 1$. Then every $x \in X$ is of the form $x = \zeta u$ with $p(x) = |\zeta|$, and for every $f \in X'$, we have $f(x) = \zeta f(u)$, $p(f) = |f(u)|$. Let g be the functional given by $g(x) = \zeta$ ($x = \zeta u \in X$). Then $S(X) = \{u, -u\}$, and Π_p has exactly two points (u, g) and $(-u, -g)$, and so $V(\hat{T})$ has exactly one point $(g(T_1 u), \dots, g(T_n u))$.

COROLLARY 3.4. Let $\hat{F} = (F_1, \dots, F_n)$ be an n -tuple of continuous mappings of $S(X)$ into X , and for each $p \in N(X)$, let $V_p(\hat{F}) = \{(f(F_1 x), \dots, f(F_n x)) : (x, f) \in \Pi_p\}$. Then $V_p(\hat{F})$ is connected, except perhaps when X has dimension one over \mathbf{R} .

For each n -tuple $\hat{T} = (T_1, \dots, T_n)$ of operators on X , the joint numerical range $W(\hat{T})$ of \hat{T} is given by

$$W(\hat{T}) = \{([T_1 x, x], \dots, [T_n x, x]) : \|x\| = 1\},$$

where $[,]$ is a consistent semi-inner product (s.i.p) of Lumer [6]. It is easy to see that $W(\hat{T}) \subset V(\hat{T}) \subset V(B(X); \hat{T})$, and $V(\hat{T})$ is the union of all numerical ranges $W(\hat{T})$ corresponding to all consistent s.i.p's on X . However, $V(\hat{T}) = W(\hat{T})$ in case of smooth space.

From the following example, we see that $W(\hat{T})$ is disconnected.

EXAMPLE 3.5. Let $X = \mathbb{C}^2$ with the sup norm $\|x\| = \max(|\zeta_1|, |\zeta_2|)$ ($x = (\zeta_1, \zeta_2) \in X$), and let $[x, y]$ be defined for $x = (\zeta_1, \zeta_2)$, $y = (\eta_1, \eta_2)$, by

$$[x, y] = \begin{cases} \zeta_1 \eta_1^* & (\text{if } |\eta_1| = \|y\|), \\ \zeta_2 \eta_2^* & (\text{if } |\eta_1| < \|y\|). \end{cases}$$

Then $[,]$ is a s. i. p on X , and $[x, x] = \|x\|^2$. Define the operator T_j by $T_j x = (\zeta_1, 0)$ ($x = (\zeta_1, \zeta_2) \in X$, $j=1, 2$). For each 2-tuple $\hat{T} = (T_1, T_2)$, we have

$W(\hat{T}) = W(T_1, T_2) = \{(0, 0), (1, 1)\}$. For if $\|x\|=1$ and $|\zeta_1|=1$, then $([T_1 x, x], [T_2 x, x]) = (\zeta_1 \zeta_1^*, \zeta_1 \zeta_1^*) = (1, 1)$, and if $\|x\|=1$ and $|\zeta_1| < 1$, then

$$([T_1 x, x], [T_2 x, x]) = (0 \zeta_2^*, 0 \zeta_2^*) = (0, 0).$$

Thus $W(T) = \{(0, 0), (1, 1)\}$ is disconnected.

The following example shows that $V(\hat{T})$ is not closed as well as $V(T)$ in single operator case.

EXAMPLE 3.6. Let $X = l^2$ and let S be defined by $S(x_1, x_2, x_3, \dots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. We consider the tensor product of operators T_j ($j=1, 2$) on the tensor product $X \otimes X$ defined by $T_1 = S \otimes I$, and $T_2 = I \otimes S$. Then $V(\hat{T}) = W(T_1, T_2) = W(S) \times W(S) = (0, 1] \times (0, 1]$, and so $V(\hat{T})$ is not closed.

Now we have the following problems.

REMARKS 3.7. (a) For what Banach space X and $\hat{T} \in L^n(X)$ is $V(\hat{T})$ a closed set?

(b) Let K be a compact simply connected subset of \mathbb{C}^n . Is there some space X and $\hat{T} \in L^n(X)$ such that $V(\hat{T}) = K$?

(c) For what Banach space X is $V(\hat{T})$ convex for every $\hat{T} \in L^n(X)$?

(d) Is $V(\hat{T})$ always simply connected for every $\hat{T} \in L^n(X)$?

The author would like to thank the KOSEF for his partially financial support, 1987-89.

References

1. F.F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras I*, Cambridge University Press, 1971.

2. _____, *Numerical ranges I*, Cambridge University Press, 1973.
3. M. Cho, *Joint spectra of operators on Banach spaces*, Glasgow Math. J. **28**(1986), 69-72.
4. A. T. Dash, *Tensor products and joint numerical range*, Proc. Amer. Math. Soc. **40**(2)(1973), 521-526.
5. P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, NY inc. Inc. 1982.
6. G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **100**(1961), 29-43.
7. V. Wrobel, *Joint spectra and joint numerical ranges for pairwise commuting operators in Banach spaces*, Glasgow Math. J. **30**(1988), 145-153.

Cheju National University
Cheju 690-121, Korea