THE ASYMPTOTIC STABILITY FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

YOUNHEE KO*

1. Introduction

The basic idea behind Liapunov's direct method involves the study of an auxiliary function along solutions to a system of differential equations, and several steps are involved in order to use this approach. First, one must construct the auxiliary function-usually called a Liapunov function -which satisfies certain properties in compliance with the theory that has been developed. Then, the system of equations itself and the derivative of the Liapunov function along solutions to the system are examined for various attributes. Among the qualitative properties of solutions that one often can investigate using this technique are stability, uniform stability, asymptotic stability, uniform asymptotic stability.

One of the goals of this paper is to improve and supplement previous theorems in the literature regarding globally asymptotically stable and globally uniformly asymptotically stable for ordinary differential equations. In particular, we concentrate on two main directions ; namely, we seek to (i) present sufficient conditions to ensure the globally uniform asymptotic stability of the zero solution of differential equation, (ii) present the examples to apply our results.

*Department of Mathematics Education Cheju National University
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2. The Basic Definitions and Notation

Notation 2.1. We concern ourselves with systems of equations

$$x' = f(t, x), \tag{E}$$

where $x \in R^n$ and R^n denotes Euclidean n-space. When discussing global results, such as global asymptotic stability, we shall always assume that $f : R^+ \times R^n \to R^n$ [$R^+ = [0, \infty)$] is continuous. On the other hand, when considering local results, we shall usually assume that $f : R^+ \times B(h) \to R^n$ [$B(h) = \{x \in R^n \mid 0 \le |x| < h, \text{for some } h > 0\}$]. On some occassions we may assume that $t \in R$, rather than $t \in R^+$. Unless otherwise stated, we shall assume that for every $(t_0, \xi), t_0 \in R^+$, the initial value problem

$$x' = f(t, x), \qquad x(t_0) = \xi \tag{I}$$

possesses a unique solution $\phi(t, t_o, \xi)$ which depends continuously on the initial data (t_0, ξ). Since it is very natural in this chapter to think of t as representing time, we shall use the symbol t_0 in (I) to represent the initial time (rather than using τ as was done earlier). Furthermore, we shall frequently use the symbol x_0 in place of ξ to represent the initial state.

Definition 2.2. A point $x_e \in R^n$ is called an **equilibrium point** of (E) (at time $t^* \in R^+$) if

$$f(t, x_e) = 0 \qquad \text{for all} \quad t \ge t^*.$$

Other terms for equilibrium point include ***stationary point, singular point, critical point***, and ***rest position***. Note that if x_e is an equilibrium point of (E) at t^*, then it is an equilibrium point at all $\tau \ge t^*$. Note also that in the case of autonomous systems

$$x' = f(x) \tag{A}$$

and in the case of T-periodic systems

$$x' = f(t,x), \qquad f(t,x) = f(t+T,x), \tag{P}$$

a point $x_e \in R^n$ is an equilibrium at some time t^* if and only if it is an equilibrium point at all times. Also note that if x_e is an equilibrium (at t^*) of (E), then the transformation $s = t - t^*$ reduces (E) to

$$dx/ds = f(s + t^*, x),$$

and x_e is an equilibrium (at s=0) of this system. For this reason, we shall henceforth assume that $t^* = 0$ in Definition 2.2 and we shall not mention t^* further. Note also that if x_e is an equilibrium point of (E), then for any $t_0 \geq 0$

$$\phi(t, t_0, x_e) = x_e \qquad \text{for all} \quad t \geq t_0,$$

i.e., x_e is a unique solution of (E) with initial data given by $\phi(t_0, t_0, x_e) = x_e$.

Definition 2.3. The equilibrium $x = 0$ of (E) is **stable** if for every $\epsilon > 0$ and any $t_0 \in R^+$ there exists a $\delta(\epsilon, t_0) > 0$ such that

$$|\phi(t, t_0, \xi)| < \epsilon \qquad \text{for all} \quad t \geq t_0$$

whenever

$$|\xi| < \delta(\epsilon, t_0).$$

Definition 2.4. The equilibrium $x = 0$ of (E) is said to be **uniformly stable** if δ is independent of t_0 in Definition 2.3, i.e., if $\delta = \delta(\epsilon)$.

Definition 2.5. The equilibrium $x = 0$ of (E) is **asymptotically stable** if

(i) it is stable, and

(ii) for every $t_0 \geq 0$ there exists an $\eta(t_0) > 0$ such that

$$\lim_{t \to \infty} \phi(t, t_0, \xi) = 0 \qquad \text{whenever} \quad |\xi| < \eta.$$

The set of all $\xi \in R^n$ such that $\phi(t, t_0, \xi) \to 0$ as $t \to \infty$ for some $t_0 \geq 0$ is called the **domain of attraction** of the equilibrium $x = 0$ of (E). Also, if for (E) condition (ii) is true, then the equilibrium $x = 0$ is said to be **attractive**.

Definion 2.6. The equilibrium $x = 0$ of (E) is **uniformly asymptotically stable** if

(i) it is uniformly stable, and

(ii) there is a $\delta_0 > 0$ such that for every $\epsilon > 0$ and for any $t_0 \in R^+$, there exists a $T(\epsilon) > 0$, independent of t_0, such that

$$|\phi(t, t_0, \xi)| < \epsilon, \qquad \text{for all} \quad t \geq t_0 + T(\epsilon)$$

$$\text{whenever} \quad |\xi| < \delta_0.$$

Definition 2.7. The equilibrium $x = 0$ of (E) is **globally asymptotically stable** if it is stable, and if every solution of (E) tends to zero as $t \to \infty$.

Definition 2.8.. The equilibrium $x = 0$ of (E) is **globally uniformly asymptotically stable** if

(i) it is uniformly stable, and

(ii) for any $\alpha > 0$ any $\epsilon > 0$, and $t_0 \in R^+$, there exists $T(\epsilon, \alpha) > 0$, independent of t_0, such that

$$\text{if } |\xi| < \alpha, \qquad \text{then } |\phi(t, t_0, \xi)| < \epsilon \qquad \text{for all} \quad t \geq t_0 + T(\epsilon, \alpha).$$

Notation 2.9. We shall present stability results for the equilibrium $x = 0$ of a system

$$x' = f(t, x). \qquad (E)$$

Such results involve the existence of real valued functions $v : D \to R$. In the case of local results (e.g., stability, asymptotic stability), we shall usually only require that $D = B(h) \subset R^n$ for some $h > 0$, or $D = R^+ \times B(h)$. On the other hand, in the case of global results (e.g., globally asymptotic stability), we have to assume that $D = R^n$ or $D = R^+ \times R^n$. Unless stated otherwise, we shall always assume that $v(t,0) = 0$ for all $t \in R^+$ [resp., $v(0) = 0$].

Now let ϕ be an arbitrary solution of (E) and consider the function $t \longmapsto v(t,\phi(t))$. If v is continuously differentiable with respect to all of its arguments, then we obtain (by the chain rule) the derivative of v with respect to t along the solutions of (E), $v'_{(E)}$, as

$$v'_{(E)}(t,\phi(t)) = \frac{\partial v}{\partial t}(t,\phi(t)) + \nabla v(t,\phi(t))^T f(t,\phi(t)).$$

Here ∇v denotes the gradient vector of v with respect to x. For a solution $\phi(t,t_0,\xi)$ of (E), we have

$$v(t,\phi(t)) = v(t_0,\xi) + \int_{t_0}^{t} v'_{(E)}(\tau,\phi(\tau,t_0,\xi))d\tau.$$

Definition 2.10. Let $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] be continuously differentiable with respect to all of its arguments and let ∇v denote the **gradient** of v with respect to x. Then $v'_{(E)} : R^+ \times R^n \to R$ [resp., $v'_{(E)} : R^+ \times B(h) \to R$] is defined by

$$\begin{aligned} v'_{(E)}(t,x) &= \frac{\partial v}{\partial t}(t,x) + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(t,x) f_i(t,x) \\ &= \frac{\partial v}{\partial t}(t,x) + \nabla v(t,x)^T f(t,x). \end{aligned} \tag{2.1}$$

We call $v'_{(E)}$ **the derivative of** v (with respect to t) **along the solutions of** (E) [or along the trajectories of (E)].

Occasionally we shall only require that v be continuous on its domain of definition and that it satisfy locally a Lipschitz condition with respect to x. In such case we call v a **Liapunov function** and we define the **upper right–hand derivative of** v **with respect to** t **along the solutions of** (E) by

$$\begin{aligned} v'_{(E)}(t,x) &= \limsup_{\theta\to 0^+} \frac{\{v(t+\theta,\phi(t+\theta,t,x)) - v(t,x)\}}{\theta} \\ &= \limsup_{\theta\to 0^+} \frac{\{v(t+\theta,x+\theta\cdot f(t,x)) - v(t,x)\}}{\theta}. \end{aligned} \tag{2.2}$$

When v is continuously differentiable, then (2.2) reduces to (2.1).

Definition 2.11. A continuous function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] is said to be **positive definite** if

(i) $\omega(0) = 0$, and

(ii) $\omega(x) > 0$ for all $x \neq 0$ [resp., $0 < |x| \leq r$ for some $r > 0$].

Definition 2.12. A continuous function $\omega : R^n \to R$ is said to be **radially unbounded** if

(i) $\omega(0) = 0$,

(ii) $\omega(x) > 0$ for all $x \in R^n - \{0\}$, and

(iii) $\omega(x) \to \infty$ as $|x| \to \infty$.

Definition 2.13. A function ω is said to be **negative definite** if $-\omega$ is a positive definite function.

Definition 2.14. A continous function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] is said to be **positive semidefinite** if

(i) $\omega(0) = 0$, and

(ii) $\omega(x) \geq 0$ for all $x \in B(r)$ and for some $r > 0$.

Definition 2.15. A function ω is said to be **negative semidefinite** if $-\omega$ is positive semidefinite.

Next, we consider the case $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$].

Definition 2.16. A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is said to be **positive definite** if there exists a positive definite function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] such that

(i) $v(t, 0) = 0$ for all $t \geq 0$, and

(ii) $v(t, x) \geq \omega(x)$ for all $t \geq 0$ and for all $x \in B(r)$

for some $r > 0$.

Definition 2.17. A continuous function $v : R^+ \times R^n \to R$ is **radially unbounded** if there exists a radially unbounded function $\omega : R^n \to R$ such that

(i) $v(t, 0) = 0$ for all $t \geq 0$, and

(ii) $v(t, x) \geq \omega(x)$ for all $t \geq 0$ and for all $x \in R^n$.

Definition 2.18. A continuous function $v : R^+ \times R^n \to R$ [resp., $v : R^+ \times B(h) \to R$] is said to be **decrescent** if there exists a positive definite function $\omega : R^n \to R$ [resp., $\omega : B(h) \to R$] such that

$$|v(t, x)| \leq \omega(x) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \text{for all} \quad x \in B(r)$$

$$\text{for some} \quad r > 0.$$

Definition 2.19. A continuous function $W : R^+ \to R^+$ is called a **wedge** if $W(0) = 0$ and W is strictly increasing on R^+.

In presenting sufficient conditions to ensure that the zero slution of (E) is globally aymptotically stable and globally uniformly aymptotically statble the following theorems are basic

Theorem 2.20 ([12, Theorem 8.5]). If there exists a continuously differentiable, positive definite function v with $v'_{(E)}(t, x) \leq 0$, if $v'_{(E)}(t, x) \leq -c(|x|)$, where $c(r)$ is continuous on $[0, h]$ and positive definite, and if $F(t, x)$ is bounded, then the solution $x(t) = 0$ of (E) is **asymptotically stable.**

Theorem 2.21 ([11, Theorem 5.9.2]). If there exists a continuously differentiable, positive definite, decrescent function v with a negative semidefinite derivative $v'_{(E)}$, then the equilibrium $x = 0$ of (E) is **uniformly stable.**

Theorem 2.22 ([11, Theorem 5.9.7]). If there exists a continuously differentiable, positive definite, decrescent, and radially unbounded function v such that $v'_{(E)}$ is negative definite for all $(t, x) \in R^+ \times R^n$, then the equilibrium $x = 0$ of (E) is **globally uniformly asymptotically stable.**

3. Main Results and Examples

Theorem 3.1. Let a function $v : R^+ \times R^n \to R$ be continuous and locally Lipschitz in $x \in R^n$ and let $\eta : R^+ \to R^+$ be a measurable function such that $\int_0^\infty \eta(s)ds = \infty$.

Suppose that there exist wedges W_1, W_2 and W_3 such that for all $t \in R^+$ and $x \in R^n$,

(i) $W_1(|x|) \leq v(t,x) \leq W_2(|x|)$ and

(ii) $v'_{(E)}(t,x) \leq -\eta(t)W_3(|x|)$,

where

$$\lim_{r\to\infty} W_1(r) = \lim_{r\to\infty} W_2(r) = \infty.$$

Then the zero solution of (E) is **uniformly stable** and **globally asymptotically stable.**

Proof. Let $\epsilon > 0$ be given. Then there exists a $\delta = \delta(\epsilon) > 0$ such that $W_2(\delta) < W_1(\epsilon)$. Let $\phi(t, t_0, x_0)$ be a solution of (E) such that $t \geq t_0 \geq 0$ and $|x(t_0)| = |x_0| < \delta$. Then we have

$$\begin{aligned} W_1(|\phi(t,t_0,x_0)|) &\leq v(t,\phi(t,t_0,x_0)) \leq v(t_0,x_0) \\ &\leq W_2(|x_0|) < W_2(\delta) < W_1(\epsilon), \end{aligned}$$

which implies that $|\phi(t,t_0,x_0)| < \epsilon$ if $t \geq t_0$ and $|x_0| < \delta$. This proves the uniform stability of the zero solution of (E).

Now we show that the domain of attraction of $x = 0$ of (E) is all of R^n. Fix $(t_0, x_0) \in R^+ \times R^n$. Then $v(t, \phi(t, t_0, x_0))$ is nonincreasing and so has a limit $\quad r \geq 0, \quad$ where $|x_0| < \alpha$ for any $\alpha > 0$.

Suppose that

$$\phi(t,t_0,x_0)) \not\longrightarrow 0 \qquad \text{as} \qquad t\to\infty.$$

Then there exists $r > 0$ such that $\lim_{t\to\infty} v(t,\phi(t,t_0,x_0)) = r$. This implies

$$r \leq v(t,\phi(t,t_0,x_0) \leq W_2(|\phi(t)|)$$

and

$$|\phi(t)| \geq W_2^{-1}(r) \qquad \text{for all } t \geq t_0.$$

By integrating v' along $\phi(t, t_0, x_0)$, we obtain

$$\begin{aligned} v(t, \phi(t, t_0, x_0)) &\leq v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s)|) ds \\ &\leq v(t_0, x_0) - W_3(W_2^{-1}(r)) \int_{t_0}^{t} \eta(s) ds \to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

which is a contradiction. Thus the proof is complete.

Example 3.2. Consider a scalar equation

$$x' = -a(t)g(x) \tag{3.1}$$

where $a : R^+ \to R^+$. Suppose that there exists a wedge W^* such that $xg(x) \geq W^*(|x|)$ for any $x \in R$, and that $\int_0^\infty a(s)da = \infty$. Then the zero solution $x = 0$ of (3.1) is **uniformly stable** and **globally asymptotically stable.**

Proof. Consider the function $v(t, x) = \frac{1}{2}x^2$. Then

$$\begin{aligned} v'_{(3.1)}(t, x) &= x(-a(t)g(x)) \\ &= -a(t)xg(x) \\ &\leq -a(t)W^*(|x|). \end{aligned}$$

Therefore, all conditions in Theorem 3.1 are satisfied. Hence the zero solution $x = 0$ of (3.1) is uniformly stable and globally asymptotically stable.

Example 3.2 revisted Consider a scalar equation

$$x' = -a(t)g(x) \tag{3.1}$$

where $a : R^+ \to R^+$ and $g : R \to R$ are continuous such that

$$xg(x) > 0 \quad \text{for any } x \in R - \{0\}.$$

Then

(1) the zero solution $x = 0$ of (3.1) is **unique to the right,**

(2) the zero solution $x = 0$ of (3.1) is **uniformly stable,** and

(3) if $\int_0^\infty a(t)dt = \infty$, then the zero solution $x = 0$ of (3.1) is **globally asymptotically stable.**

Proof. (1) If $x > 0$, then $|x|' = x' = -a(t)g(t) \le 0$. If $x < 0$, then

$$|x|' = (-x)' = -x' = a(t)g(x) \le 0.$$

That is, $|x|' \le 0$ for all $x \in R - \{0\}$. Thus $|\phi(t)|$ is nonincreasing for any solution $\phi(t)$ of (3.1). Therefore, $\phi(t) = 0$ for all $t \ge t_0$ if there exists $t_0 \ge 0$ such that $\phi(t_0) = 0$.

(2) Let $\epsilon > 0$ be given. Then $|\phi(t, t_0, x_0)| \le |x_0| < \epsilon$ if $t_0 \ge 0$, $t \ge t_0$ and $|x_0| = |x(t_0)| = |\phi(t_0)| < \epsilon$. Put $\delta = \epsilon$. Then the zero solution $x = 0$ of (3.1) is uniformly stable.

(3) Let $\phi(t, t_0, x_0)$ be a solution of (3.1). Then $\phi(t, 0, x_0) \ge 0$ for all $t \ge 0$ if $\phi(0) = x_0 \ge 0$, and $\phi(t, 0, x_0) \le 0$ for all $t \ge 0$ if $\phi(0) = x_0 \le 0$, since the zero solution of (3.1) is unique to the right.

Case 1). Let $\phi(0) = x_0 \ge 0$. Then $\phi(t, 0, x_0)$ is nonincreasing, since $|\phi(t)|' \le 0$ for any $t \ge 0$.

Now we claim that $\lim_{t\to\infty} \phi(t) = 0$. Suppose not. Then there is a $p > 0$ such that $\lim_{t\to\infty} \phi(t) = p$. By assumption on g $1/g(x)$ is bounded for all $x \in (p, \phi(0))$. Thus we have

$$\int_p^{\phi(0)} \frac{1}{g(x)} dx = \int_\infty^0 -a(t)dt = \int_0^\infty a(t)dt = \infty,$$

which is a contradiction.

Case 2). Let $\phi(0) = x_0 < 0$. Then $\phi(t, 0, x_0)$ is nondecreasing, since $|\phi(t)|' \le 0$ for any $t \ge 0$.

Now we claim that $\lim_{t\to\infty} \phi(t) = 0$. Suppose not. Then there is a $q < 0$ such that $\lim_{t\to\infty} \phi(t) = q$. By assumption on g $1/g(x)$ is bounded for all $x \in (\phi(0), q)$. Thus we have

$$\int_{\phi(0)}^q \frac{1}{g(x)} dx = \int_0^\infty -a(t)dt = -\int_0^\infty a(t)dx = -\infty,$$

which is a contradiction. Hence the proof is complete.

In the process of the above proof of Example 3.2 revisted we do not use the result of Theorem 3.1. Suppose that we replace the condition that $xg(x) > 0$ for any $x \in R - \{0\}$ with the condition that $xg(x) \geq W^*(|x|)$ for some wedge W^* and any $x \in R$ in Example 3.2 revisted. Then we apply Theorem 3.1 to prove the Example 3.2 revisted.

Remark 3.3. We can easily find a function $g(x)$ such that $xg(x) > 0$ for $x \in R - \{0\}$ implies that there exists a wedge W^* such that $W^*(|x|) \leq g(x)x$ for any $x \in R$. Consider $g(x) = Mx^n$, where $M > 0$ and n is a positive odd number. Then $xg(x) = Mx^{n+1} > 0$ if $x \in R - \{0\}$. Furthermore, we can consider $W^*(|x|) = xg(x) = Mx^{n+1}$.

Remark 3.4. In practical it is almost imposible that we find the example which satisfies the condition $v'_{(E)}(t,x) \leq -c(|x|)$ without satisfying the condition $v(t,x) \leq W(|x|)$ for some wages c, W. Therefore Theorem 3.1 generalizes partially Theorem 2.20. Also Theorem 3.1 generalizes Theorem 2.21 since if η is a constant function, then $\int_0^\infty \eta(s)ds = \infty$.

Lemma 3.5. Let $\eta : R^+ \to R^+$ be a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$. Then for any $M > 0$ there exists a $\delta = \delta(M) > 0$ such that

$$\int_t^{t+\delta} \eta(s)ds > M \qquad \text{for any} \quad t \in R^+.$$

Proof. Suppose that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$. Then for 1 there exists $\delta_0 > 0$ such that $\int_t^{t+\delta_0} \eta(s)ds > 1$ for any $t \in R^+$.

Let $M > 0$ be given. If $M \leq 1$, then we may take δ as $\delta = \delta_0$. if $M > 1$, then there exists a positive integer N with $M \leq N$. Thus we have

$$M \le N < \int_t^{t+\delta_0} \eta(s)ds + \int_{t+\delta_0}^{t+2\delta_0} \eta(s)ds + \cdots + \int_{t+(N-1)\delta_0}^{t+N\delta_0} \eta(s)ds$$
$$= \int_t^{t+N\delta_0} \eta(s)ds$$

for any $t \in R$. That is, $\delta = N\delta_0$. Hence the proof is complete.

Theorem 3.6. Let a function $v : R^+ \times R^n \to R$ be continuous and locally Lipschitz in $x \in R^n$ and let $\eta : R^+ \to R^+$ be a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$.

Suppose that there exist wedges W_1, W_2 and W_3 such that for all $t \in R^+$ and $x \in R^n$,

(i) $W_1(|x|) \le v(t,x) \le W_2(|x|)$ and

(ii) $v'_{(E)}(t,x) \le -\eta(t)W_3(|x|)$,

where

$$\lim_{r\to\infty} W_1(r) = \lim_{r\to\infty} W_2(r) = \infty.$$

Then the zero solution of (E) is **globally uniformly asymptotically stable.**

Proof. Let $\epsilon > 0$ be given. Then there exists a $\delta = \delta(\epsilon) > 0$ such that $W_2(\delta) < W_1(\epsilon)$. Let $\phi(t, t_0, x_0)$ be a solution of (E) such that $t \ge t_0 \ge 0$ and $|x(t_0)| = |x_0| < \delta$. Then we have

$$W_1(|\phi(t,t_0,x_0)|) \le v(t,\phi(t,t_0,x_0)) \le v(t_0,x_0)$$
$$\le W_2(|x_0|) < W_2(\delta) < W_1(\epsilon),$$

which implies that $|\phi(t,t_0,x_0)| < \epsilon$ if $t \ge t_0$ and $|x_0| < \delta$. This proves the uniform stability of the zero solution of (E).

For 1 take $\delta_0 = \delta_0(1)$ of the uniform stability. By the Lemma 3.5 there is an $L = L(\epsilon)$ such that

$$\int_{t_1}^{t_1+L} \eta(s)ds > W_2(\delta_0)/W_3(\delta) \quad \text{for all} \quad t_1 \in R^+.$$

Let $\phi(t, t_0, x_0)$ be a solution of (E) with $|x_0| < \delta_0$. Now we claim that $|\phi(t^*, t_0, x_0)| < \delta$ for some $t^* \in [t_0, t_0 + L]$. For if this were not true, we would have

$$|\phi(t, t_0, x_0)| \geq \delta \qquad \text{for all} \quad t \in [t_0, t_0 + L].$$

Thus

$$\begin{aligned}
0 < W_1(\delta) \leq v(t, \phi(t, t_0, x_0)) &\leq v(t_0, x_0) + \int_{t_0}^{t} v'_{(E)}(s, \phi(s))ds \\
&\leq v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s)|)ds \\
&< W_2(\delta_0) - W_3(\delta) \int_{t_0}^{t} \eta(s)ds \\
&< W_2(\delta_0) - W_3(\delta) W_2(\delta_0)/W_3(\delta) = 0,
\end{aligned}$$

which is a contradiction if $t = t_0 + L$. Therefore, for $t \geq t_0 + L$ and some $t^* \in [t_0, t_0 + L]$ with $|\phi(t^*)| < \delta$,

$$\begin{aligned}
W_1(|\phi(t)|) &\leq v(t, \phi(t)) \leq v(t^*, \phi(t^*)) \\
&\leq W_2(|\phi(t^*)|) < W_2(\delta) < W_1(\epsilon),
\end{aligned}$$

which implies that the zero solution of (E) is uniformly asymptotically stable.

Finally, we show that the domain of attraction of $x = 0$ of (E) is all of R^n. Fix $(t_0, x_0) \in R^+ \times R^n$. Then $v(t, \phi(t, t_0, x_0))$ is nonincreasing and so has a limit $a \geq 0$, where $|x_0| < \alpha$ for any $\alpha > 0$.

Suppose that no $T = T(\alpha, \epsilon) > 0$ exists. Then

$$\lim_{t \to \infty} v(t, \phi(t, t_0, x_0)) = a > 0 \qquad \text{for some } x_0 \text{ with } |x_0| < \alpha.$$

Thus, for $t \geq t_0$ we have

$$\begin{aligned}
a &\leq v(t, \phi(t, t_0, x_0)) \\
&\leq v(t_0, x_0) - \int_{t_0}^{t} \eta(s) W_3(|\phi(s, t_0, x_0)|)ds \\
&\leq v(t_0, x_0) - W_3(W_2^{-1}(a)) \int_{t_0}^{t} \eta(s)ds.
\end{aligned}$$

Therefore, the right-hand side of this inequality becomes negative for t sufficiently large. But this is impossible when $a > 0$. Hence the proof is complete.

Remark 3.7. Theorem 3.6 generalizes Theorem 2.22 For if η is a constant, then $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in R^+$. Furthermore, if v is continuously differentiable, then v is continuous and locally Lipschitz in $x \in R^n$.

Remark 3.8. Consider a scalar differential equation

$$x' = -\frac{1}{t+1}x \qquad \text{on } [0,\infty). \tag{3.2}$$

Then the zero solution $x = 0$ of (3.2) is **uniformly stable** and **globally asymptotically stable** (by Theorem 3.1). But it is well known that the zero solution of (3.2) is not globally uniformly asymptotically stable. In fact, $\eta(t) = \frac{1}{t+1}$ does not satisfy the condition in Theorem 3.6

Example 3.9. Consider a scalar differential equation

$$x' = -|\text{sin}t|x^n \qquad \text{on } [0,\infty), \tag{3.3}$$

where n is a positive odd integer. Then the zero solution of (3.3) is **globally uniformly asymptotically stable.**

Proof. Consider the function $v(t,x) = \frac{1}{2}x^2$. Then

$$\begin{aligned} v'_{(3.3)}(t,x(t)) &= xx' = x(-|\text{sin}t|x^n) \\ &= -|\text{sin}t|x^{n+1}. \end{aligned}$$

Let $\eta(t) = |\text{sin}t|, W_1(t) = \frac{1}{3}t^2, W_2(t) = t^2$ and $W_3(t) = \frac{1}{2}t^{n+1}$. Then all conditions in Theorem 3.6 are satisfied. Hence the zero solution of (3.3) is globally uniformly asymptotically stable.

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