Acta Sci. Math. (Szeged)
59 (1994), 267–278

On the uniform asymptotic stability for functional differential equations

YOUNHEE KO

1. Introduction

The purpose of this paper is to present some uniform asymptotic stability theorems for functional differential equations

We consider a system of functional differential equations with finite delay. Also we consider a system of functional differential equations with unbounded delay. For $x \in \mathbf{R}^n$ with $x = (x_1, \ldots, x_n)$, $|x|$ denotes a usual norm in $\mathbf{R}^n$, and W_i denotes a continuous function from $\mathbf{R}_+$ into $\mathbf{R}_+$ such that $W_i(0) = 0$ and W_i is strictly increasing on $\mathbf{R}_+ = [0, \infty)$. Our main goal is to generalize two theorems in [5, Theorem 8-(d)] and [8, Theorem 2.1].

2. Uniform asymptotic stability I

In this section we consider the system

$$x'(t) = F(t, x_t), \tag{1}$$

where x_t is the translation of x on $[t-h, t]$ back to $[-h, 0]$, where $h > 0$ is a fixed constant, and x' denotes the right-hand derivative. The following notations will be used.

For $h > 0, C$ denotes the space of continuous functions mapping $[-h, 0]$ into $\mathbf{R}^n$, and for $\phi \in C, \|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$. Also, C_H denotes the set of $\phi \in C$ with $\|\phi\| < H$. If x is a continuous function of u defined for $-h \le u < A$, with $A > 0$, and if t is a fixed number satisfying $0 \le t < A$, then x_t denotes the translation of the restriction of x to $[t-h, t]$ so that x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$ for $-h \le \theta \le 0$. We denote by $x(t_0, \phi)$ a solution of (1) with

Received February 12, 1992 and in revised form February 8, 1994.
AMS Subject Classification (1991): 34K20, 34D20.

the initial condition $x_{t_0}(t_0, \phi) = \phi \in C$ and we denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t.

It is supposed that $F: \mathbf{R}_+ \times C_H \longrightarrow \mathbf{R}^n$ is continuous and takes bounded sets into bounded sets; where $0 < H \leq \infty$. It is well known ([7], [10]) that for each $t_0 \in \mathbf{R}_+ = [0, \infty)$ and each $\phi \in C_H$ there is at least one solution $x(t_0, \phi)$ defined on an interval $[t_0, t_0 + \alpha)$ and, if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$, then $\alpha = \infty$.

A Liapunov functional is a continuous function $V(t, \phi): \mathbf{R}_+ \times C_H \longrightarrow \mathbf{R}_+$ whose derivative along a solution of (1) satisfies some specific relation. The derivative of a Liapunov functional $V(t, \phi)$ along a solution $x(t)$ of (1) may be defined by

$$V'_{(1)}(t, \phi) = \limsup_{\delta \to 0^+} \{V(t + \delta, x_{t+\delta}(t, \phi)) - V(t, \phi)\}/\delta.$$

Definition 2.1. Let $F(t, 0) = 0$, for all $t \geq 0$.

(a) The zero solution of (1) is said to be *stable* if for each $\epsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \epsilon$.

(b) The zero solution of (1) is *uniformly stable* (U. S.) if it is stable and if δ is independent of t_0.

(c) The zero solution of (1) is *asymptotically stable* (A. S.) if it is stable and if for each $t_0 \geq 0$ there is a $\sigma > 0$ such that $\phi \in C_\sigma$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

(d) The zero solution of (1) is *uniformly asymptotically stable* (U. A. S) if it is U. S. and if there is an $\eta > 0$ and for each $\gamma > 0$ there exists $T > 0$ such that $[t_0 \in \mathbf{R}_+, \phi \in C_\eta, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \gamma$.

Theorem 2.1. *Suppose that $D, V: \mathbf{R}_+ \times C_H \longrightarrow \mathbf{R}_+$ are continuous, $\eta: \mathbf{R}_+ \to \mathbf{R}_+$ is a measurable function such that $\lim_{s\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$, and the following conditions are satisfied:*

(i) $W_1(|x(t)|) \leq V(t, x_t) \leq W_2(D(t, x_t))$;

(ii) $V'_{(1)}(t, x_t) \leq -\eta(t) W_3(D(t, x_t))$;

(iii) $D(t, x_t) \leq W_4(\|x_t\|)$.

Then $x = 0$ of (1) is U. A. S. .

Proof. There is a W_5 with $V(t, \phi) \leq W_5(\|\phi\|)$ so $x = 0$ of (1) is U. S. . Let $\delta > 0$ correspond to $K \in (0, H)$ in the sense of the definition of U. S. and let $\epsilon > 0$ be given. We try to find T such that $[t_0 \in \mathbf{R}_+, \|\phi\| < \delta, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \epsilon$.

Find $\xi = \xi(\epsilon) > 0$ with $W_2(\xi) < W_1(\epsilon)$. First we prove the existence of an $L = L(\epsilon) > 0$ such that for each $t_1 \geq t_0$ the interval $[t_1, t_1 + L]$ contains a point t^* with $D(t^*, x_{t^*}) < \xi$.

By assumption on η there is an $L = L(\epsilon)$ such that

$$\int_{t_1}^{t_1+L} \eta(s)ds > W_5(\delta)/W_3(\xi) \qquad \text{for all } t_1 \in \mathbf{R}_+.$$

If $D(t, x_t) \geq \xi$ were true for all $t \in [t_1, t_1 + L]$, then we would have

$$\begin{aligned} 0 &\leq V(t_1 + L, x_{t_1+L}) \leq V(t_0, \phi) - \int_{t_1}^{t_1+L} \eta(s) W_3(D(s, x_s))ds \\ &< W_5(\delta) - W_3(\xi) \int_{t_1}^{t_1+L} \eta(s) < 0, \end{aligned}$$

which is a contradiction. Therefore, for $t \geq t_0 + L$ and some $t^* \in [t_0, t_0 + L]$,

$$\begin{aligned} W_1(|x(t)|) &\leq V(t, x_t) \leq V(t^*, x_{t^*}) \\ &\leq W_2(D(t^*, x_{t^*})) < W_2(\xi) < W_1(\epsilon), \end{aligned}$$

which completes the proof.

■

Example 2.1. *Consider the scalar equation*

$$\text{(A)} \qquad x'(t) = -a(t)x(t) + b(t)x(t) \int_{-h}^{0} x^2(t+s)ds,$$

where $a: \mathbf{R}_+ \to \mathbf{R}_+$ *is continuous such that* $\lim_{S\to\infty} \int_t^{t+S} a(s)ds = \infty$ *uniformly with respect to* $t \in \mathbf{R}_+$, *and* $b: \mathbf{R}_+ \to \mathbf{R}_+$ *is continuous. Then* $x = 0$ *of* (A) *is U. A. S. .*

Proof. Let $W_1(t) = \frac{1}{2}t^2, W_2(t) = t, W_3(t) = t, W_4(t) = t^2, V(t, x_t) = \frac{1}{2}x^2(t)$, and $D(t, x_t) = x^2(t)$. Then

$$\frac{1}{2}x^2(t) = W_1(|x(t)|) = V(t, x_t) \leq W_2(D(t, x_t)) = x^2(t),$$

$$\begin{aligned} V'(t, x_t) &= x(t)x'(t) = -a(t)x^2(t) - x^2(t)b(t) \int_{-h}^{0} x^2(t+s)ds \\ &\leq -a(t)x^2(t) = -a(t)W_3(D(t, x_t)), \end{aligned}$$

and

$$D(t, x_t) = x^2(t) \leq \|x_t\|^2 = W_4(\|x_t\|).$$

Thus $x = 0$ of (A) is U. A. S. .

■

Example 2.2. *Consider the scalar equation*

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} x(s)ds, \tag{B}$$

where $a\colon \mathbf{R}_+ \to \mathbf{R}_+$, $b\colon \mathbf{R}_+ \to \mathbf{R}$ *are continuous;* $\eta(t) = a(t) - a\int_t^{t+h}|b(s)|ds \geq 0$ *for some* $\alpha > 1$, $\lim_{S\to\infty}\int_t^{t+S}\eta(s)ds = \infty$ *uniformly with respect to* $t \in \mathbf{R}_+$, $\int_t^{t+h}|b(s)|ds \leq M$ *for some* $M > 0$ *and any* $t \in \mathbf{R}_+$, *and* $|b(t)| \geq (\frac{M\alpha}{\alpha-1})\eta(t)$. *Then* $x+0$ *of* (B) *is U. A. S.* .

Proof. Consider the functionals

$$V(t,x_t) = |x(t)| + \alpha\int_{-h}^{0}\int_{t+s}^{t}|b(u-s)||x(u)|duds$$

and

$$D(t,x_t) = |x(t)| + M\alpha\int_{t-h}^{t}|x(u)|du.$$

Then we have

$$\begin{aligned} V(t,x_t) &= |x(t)| + \alpha\int_{-h}^{0}\int_{t+s}^{t}|b(u-s)||x(u)|duds \\ &= |x(t)| + \alpha\int_{t-h}^{t}[\int_{t}^{u+h}|b(s)|ds]|x(u)|du \\ &\leq |x(t)| + \alpha\int_{t-h}^{t}[\int_{u}^{u+h}|b(s)|]|x(u)|duds \\ &\leq |x(t)| + M\alpha\int_{t-h}^{t}|x(u)|du = D(t,x_t) \end{aligned}$$

and

$$D(t,x_t) = |x(t)| + M\alpha\int_{t-h}^{t}|x(u)|du \leq (1+hM\alpha)\|x_t\|.$$

Furthermore,

$$\begin{aligned} V'(t,x_t) &\leq -a(t)|x(t)| + b(t)|\int_{t-h}^{t}|x(s)|ds + \alpha\int_{-h}^{0}|b(t-s)|ds|x(t)| \\ &\quad - \alpha\int_{-h}^{0}|b(t)||x(t+s)|ds = -\eta(t)|x(t)| + |b(t)|(1-\alpha)\int_{t-h}^{t}|x(s)|ds \\ &= -\eta(t)D(t,x_t) + \{M\alpha\eta(t) + |b(t)|(1-\alpha)\}\int_{t-h}^{t}|x(u)|du \\ &\leq -\eta(t)D(t,x_t). \end{aligned}$$

Therefore, $x = 0$ of (B) is U. A. S. . ∎

In particular, $||x_t||$ and $|||x_t||| := [\int_{-h}^{0} x^2(t+s)ds]^{1/2}$ can be choosen in the capacity of $D(t, x_t)$ in Theorem 2.1.

Remark 2.1. By conditions (i) and (ii), for the Lyapunov functional V in Theorem 2.1 the inequality

$$V'_{(1)}(t, x_t) \le -\eta(t) W_3(W_3(W_2^{-1}(V(t, x_t)))$$

holds. This makes possible to prove the theorem by the method of differential inequalities (see e.g. [9]).

3. Uniform asymptotic stability II

In this section we consider a system of functional differential equations

(2) $$x'(t) = F(t, x_t),$$

where x_t represents a function from $[\alpha, t] \longrightarrow \mathbf{R}^n$, $-\infty \le \alpha \le t_0$, and F is defined for $t \ge t_0$.

For any $t \ge t_0$, by

$$(X(t), ||\cdot||)$$

we shall mean the space of continuous bounded functions $\phi: [\alpha, t] \to \mathbf{R}^n$ with $||\phi|| = \sup_{\alpha \le s \le t} |\phi(s)|$. The symbol $X_H(t)$ denotes those $\phi \in X(t)$ with $||\phi|| \le H$. An element $\phi \in X(t)$ will often be denoted by ϕ_t. In particular, $||\phi_t||$ indicates that $\phi \in X(t)$ and that the supremum is taken over the interval $[\alpha, t]$. We emphasize that ϕ_t is not shifted to the interval $[\alpha, t]$.

It is supposed that $F(t, x_t)$ is a continuous function of $t_0 \le t < \infty$ whenever $x_t \in X_H(t)$ for $t_0 \le t < \infty$. In addition, it is assumed that whenever $\psi \in X_H(t)$ and $\{\psi^{(n)}\}$ is a sequence in $X_H(t)$ with $||\psi^{(n)} - \psi|| \to 0$ as $n \to \infty$, then $F(t, \psi^{(n)}) \to F(t, \psi)$ as $n \to \infty$. It is supposed that F takes closed bounded sets of $\mathbf{R} \times X(t)$ into bounded sets of $\mathbf{R}^n$.

A solution to the initial value problem is defined in the usual manner and the reader is referred to Driver [6] or Burton [2,3] on the subjects of existence, uniqueness, and continuability of solutions.

Definition 3.1. Let $F(t, 0) = 0$. The zero solution of (2) is

(a) *stable* if for each $t_0 \in \mathbf{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that $[\phi \in X_\delta(t_0), t \geq t_0]$ imply that $|x(t, t_0, \phi)| \leq \epsilon$;

(b) *uniformly stable* (U. S.) if it is stable and if δ is independent of t_0;

(c) *asymptotically stable* (A. S.) if it is stable and for $t_0 \in \mathbf{R}$ there is an $\eta > 0$ such that $\phi \in X_\eta(t_0)$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$;

(d) *uniformly asymptotically stable* (U. A. S.) if it is uniformly stable and if there is an $\eta > 0$ and for each $\mu > 0$ there exists $T > 0$ such that $[t_0 \in \mathbf{R}, \phi \in X_\eta(t_0), t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \mu$.

Definition 3.2. Let $h > 0$ be given. A measurable function $\eta: \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is said to be *positive in measure* (PIM) with h if for every $\epsilon > 0$ there are $T \in \mathbf{R}_+$ and $\delta > 0$ such that $[t \geq T, Q \subset [t-h, t]$ is open, $\mu(Q) \geq \epsilon]$ imply that $\int_Q \eta(t)dt \geq \delta$. (Here $\mu(Q)$ denotes the Lebesgue measure of Q.)

Lemma 3.1. *Let $\{x_n\}$ be a sequence of continuous functions with continuous derivatives, $x_n: [0, 1] \longrightarrow [0, 1]$. Let $g: [0, \infty) \to [0, \infty)$ be continuous, $g(0) = 0$, $g(r) > 0$ for $r > 0$ and let g be nondecreasing. If there exists $\nu > 0$ with $\int_0^1 x_n(t)dt \geq \nu$ for all n, then there exists $\beta > 0$ with $\int_0^1 g(x_n(t))dt \geq \beta$ for all n.*

Proof. See [1, Lemma]. ■

Lemma 3.2. *Let $K > 0$ be given and suppose that η is* PIM *with h. Then for each wedge W_1 and $\alpha > 0$ there are $\beta > 0$ and $T \in \mathbf{R}_+$ such that if $f: \mathbf{R}_+ \longrightarrow \mathbf{R}$ is measurable, $f^2(s) \leq K$ for $s \in \mathbf{R}_+, t \geq T, \int_{t-h}^t f^2(s)ds \geq \alpha$, then $\int_{t-h}^t \eta(s)W_1(|f(s)|ds \geq \beta$.*

Proof. See [4, Lemma 2]. ■

Definition 3.3. A measurable function $\eta: \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ is said to be *integrally positive with parameter $\delta > 0$* (IP(δ)) if whenever $I = \cup_{m=1}^{\infty}[\alpha_m, \beta_m]$ with $\alpha_m < \beta_m < \alpha_{m+1}$ and $\beta_m - \alpha_m \geq \delta$ ($m = 1, 2, 3, \ldots$), then $\int_I \eta(t)dt = \infty$. If a function $\eta(t)$ is integrally positive for every $\delta > 0$ then it is called *integrally positive* (IP).

Lemma 3.3. *Let $\eta: \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ be measurable.*

(i) *If η is integrally positive with parameter $\delta > 0$, then $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$.*

(ii) *If $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$, then η is not necessarily integrally positive.*

Proof. See [11, Lemma 2.2, Remark 2.1]. ■

Lemma 3.4. *If $\eta(t)$ is* PIM, *then for any $K > 0$ there exists $M = M(K) > 0$ such that*

$$\int_t^{t+M} \eta(s)ds \geq K \text{ for any } t \in \mathbf{R}_+,$$

Proof. By Lemma 3.3 and Theorem 11 in [4]. ■

Theorem 3.1. *Let V be continuous and locally Lipschitz for $t_0 \leq t < \infty$ and $x_t \in X_H(t)$. Suppose that function $\Phi: [0,\infty) \longrightarrow [0,\infty)$ is continuous and integrable on $[0,\infty)$. If*

(i) $W_1(|x(t)|) \leq V(t,x_t) \leq W_2(|x(t)|) + W_3(\int_\alpha^t \Phi(t-s)W_4(|x(s)|)ds)$

and

(ii) $V'_{(1)}(t,x_t) \leq -\eta(t)W_5(|x(t)|)$,

where $\eta(t)$ is PIM *with arbitrary $h > 0$, then $x = 0$ of (2) is uniformly asymptotically stable.*

Proof. Let $\epsilon > 0$ be given. If $t_0 \geq 0$ and $\phi \in X(t_0)$ with $||\phi|| < \delta$ and $0 < \delta < \epsilon$, then for $x(t,t_0,\phi) = x(t)$ we have

$$\begin{aligned} W_1(|x(t)|) \leq V(t,x_t) \leq V(t_0,\phi) &\leq W_2(\delta) + W_3\{W_4(\delta)\int_0^{t_0-\alpha} \Phi(u)du\} \\ &\leq W_2(\delta) + W_3\{W_4(\delta)\int_0^\infty \Phi(u)du\} \leq W_1(\epsilon) \end{aligned}$$

if δ is small enough. This proves the uniform stability.

For $\epsilon > 0$ take $\delta = \delta(\epsilon)$ of U. S. . Let ν be given and find $\theta = \theta(\nu) > 0$ with $W_2(\theta) + W_3(2\theta) < W_1(\nu)$. We try to find $T = T(\nu) > 0$ such that $[t_0 \geq 0, ||\phi|| < \delta, t \geq t_0 + T]$ imply that

$$|x(t,t_0,\phi)| < \nu.$$

Now, find $r = r(\nu) > 1$ with

$$W_4(\epsilon)\int_r^\infty \Phi(u)du < \theta.$$

If $\phi \in X_\delta(t_0), t_0 \geq 0, t \geq t_0 + r$, then for $x(t) = x(t, t_0, \phi)$ we have

$$\begin{aligned}
W_1(|x(t)|) \leq V(t, x_t) &\leq W_2(|x(t)|) + W_3\{\int_\alpha^t \Phi(t-s)W_4(|x(s)|)ds\} \\
&\leq W_2(|x(t)|) + W_3\{W_4(\epsilon)\int_r^{t-\alpha} \Phi(u)du + \int_{t-r}^t \Phi(t-s)W_4(|x(s)|)ds\} \\
&\leq W_2(|x(t)|) + W_3\{W_4(\epsilon)\int_r^\infty \Phi(u)du + \\
&\qquad + (\int_0^r \Phi^2(u)du)^{1/2}(\int_{t-r}^t (W_4)^2(|x(s)|)ds)^{1/2}\} \\
&\leq W_2(|x(t)|) + W_3\{\theta + \sqrt{M}(\int_{t-r}^t (W_4)^2(|x(s)|)ds)^{1/2}\},
\end{aligned}$$

where $\int_0^r \Phi^2(u)du \leq M$ for some $M > 0$.

Now we prove the existence of an $L = L(\nu) > 0$ such that for each $t \geq t_0 \geq 0$ the interval $[t, t+L]$ contains a point t^* with $|x(t^*)| < \theta$.

By assumption on $\eta(t)$ and Lemma 3.4 there is an $L = L(\nu)$ such that

$$\int_t^{t+L} \eta(s)ds > \{W_2(\delta) + W_3(KW_4(\delta))\}/W_5(\theta) \quad \text{for all } t \geq t_0 \geq 0,$$

where

$$\int_0^\infty \Phi(s)ds < K \quad \text{for some } K > 0.$$

If $|x(t)| \geq \theta$ were true for all $t \in [t_1, t_1 + L]$ with $t_1 \geq t_0$, then we would have

$$\begin{aligned}
0 \leq V(t_1 + L, x_{t_1+L}) &\leq V(t_0, \phi) - \int_{t_1}^{t_1+L} \eta(s)W_5(|x(s)|)ds \\
&\leq W_2(\delta) + W_3(W_4(\delta)K) - W_5(\theta)\int_{t_1}^{t_1+L} \eta(s)ds < 0,
\end{aligned}$$

a contradiction.

Consider the intervals

$$I_j = [t_0 + jL, t_0 + (j+1)L], \qquad j = 1, 2, 3, \ldots$$

and find $t_j \in I_j$ with $|x(t_j)| < \theta$. Now consider a sequence of functions $\{x_k(t)\}$ defined by $x_k(t) = x(t, t_0, \phi)$ for $t_k - r \le t \le t_k$. We may take L with $L > r$. Suppose now for a moment that

$$\int_{t_k - r}^{t_k} (W_4)^2(|x(s)|)ds \ge \theta^2/M$$

for all k. Then by Lemma 3.1 and Lemma 3.2 there are $N_1 = N_1(\nu)$ and $\beta = \beta(\nu) > 0$ such that

$$\int_{t_k - r}^{t_k} \eta(s) W_5(|x(s)|)ds \ge \beta \qquad \text{for any} \quad k \ge N_1.$$

Thus we have

$$\begin{aligned} V(t, x_t) &\le V(t_0, \phi) - \int_{t_0}^{t} \eta(s) W_5(|x(s)|)ds \\ &\le W_2(\delta) + W_3(W_4(\delta)K) - \sum_{i=1}^{N} \int_{t_{2i-1}-r}^{t_{2i-1}} \eta(s) W_5(|x(s)|)ds \\ &\le W_2(\delta) + W_3(W_4(\delta)K) - (N - N_1)\beta < 0 \end{aligned}$$

if

$$N - N_1 > \{W_2(\delta) + W_3(W_4(\delta)K)\}/\beta,$$

which is a contradiction. Thus we can choose $N_2 = N_2(\nu)$ as the smallest positive integer greater than

$$\{W_2(\delta) + W_3(KW_4(\delta))\}/\beta + N_1.$$

Thus we have $|x(t_n, t_0, \phi)| < \theta$ and

$$\int_{t_n - r}^{t_n} (W_4)^2(|x(s)|)ds < \theta^2/M \qquad \text{for some} \quad n \le N_2.$$

Therefore, if $T = 2N_2L$ then $t \ge t_0 + T$ implies

$$\begin{aligned} W_1(|x(t)|) &\le V(t, x_t) \le V(t_n, x_{t_n}) \\ &\le W_2(|x(t_n)|) + W_3(\theta + \sqrt{M}(\theta/\sqrt{M})) \\ &\le W_2(\theta) + W_3(2\theta) < W_1(\nu), \end{aligned}$$

which completes the proof.

∎

Remark 3.1. The above theorem does not require that $\Phi(t)$ is bounded on $[0,\infty)$. That is, the above theorem generalizes Theorem 8-(d) in [5].

Theorem 3.2. *Let D and V be continuous and locally Lipschitz for $x_t \in X_H(t)$. Suppose that $\eta: \mathbf{R}_+ \to \mathbf{R}_+$ is a measurable function such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$ and the following conditions are satisfied:*

(i) $W_1(|x(t)|) \le V(t,x_t) \le W_2(D(t,x_t))$;

(ii) $V'_{(2)}(t,x_t) \le -\eta(t)W_3(D(t,x_t))$;

(iii) $D(t,x_t) \le W_4(\|x_t\|)$.

Then $x=0$ of (2) is U. A. S. .

Proof. See the proof of Theorem 2.1. ∎

Example 3.1. *Consider a scalar equation*

$$x'(t) = -a(t)x(t) + \alpha \int_0^t c(t-s)x(s)ds, \tag{C}$$

where $a, c: \mathbf{R}_+ \to \mathbf{R}$ are continuous, $a(t) \ge 0$ for $t \ge 0$, $\alpha > 0$ is a constant. Suppose

(i) $M := \int_0^\infty |c(v)|dv < \infty$; $\int_t^\infty |c(v)dv \le K|c(t)|$ for some $K > 0$;

(ii) there exists a constant $\beta > 0$ such that $\lim_{S\to\infty} \int_t^{t+S} \eta(s)ds = \infty$ uniformly with respect to $t \in \mathbf{R}_+$ and $(\alpha-\beta)+\beta K\eta(t) \le 0$, where $\eta(t) = a(t) - \beta M$.

Then $x=0$ of (C) is U. A. S. .

Proof. Consider the functional

$$V(t,x_t) = |x(t)| + \beta \int_0^t \int_t^\infty |c(u-s)du|x(s)ds = D(t,x_t).$$

Then we have

$$\begin{aligned} D(t,x_t) &\le \|x_t\| + \beta\|x_t\| \int_0^t \int_t^\infty |c(u-s)|duds \\ &\le \|x_t\| + \beta L\|x_t\| \end{aligned}$$

for some $L > 0$ and

$$
\begin{aligned}
V'_{(C)}(t,x_t) &\le -a(t)|x(t)| + \alpha \int_0^t |c(t-s)||x(s)|ds \\
&\quad + \beta \int_t^\infty |c(u-t)|du|x(t)| - \beta \int_0^t |c(t-s)|x(s)|ds \\
&\le (-a(t)+\beta M)|x(t)| + (\alpha-\beta)\int_0^t |c(t-s)||x(s)|ds \\
&= -\eta(t)\{D(t,x_t) - \beta \int_0^t \int_t^\infty |c(u-s)|du|x(s)|ds\} \\
&\quad + (\alpha-\beta)\int_0^t |c(t-s)||x(s)|ds \le -\eta(t)D(t,x_t) \\
&\quad + \beta K \eta(t) \int_0^t |c(t-s)||x(s)|ds + (\alpha-\beta)\int_0^t |c(t-s)||x(s)|ds \\
&\le -\eta(t)D(t,x_t).
\end{aligned}
$$

Then $x = 0$ of (C) is U. A. S. ∎

Remark 3.2. By Theorem 3.1 and using the functional

$$V(t,x_t) = |x(t)| + \alpha \int_0^t \int_t^\infty (u-s)|du|x(s)|ds$$

we can also show that the zero solution of (C) is U. A. S. under the condition that $\eta(t) = a(t) - \alpha M$ is positive in measure, $\int_0^\infty |c(u)|du \le M < \infty$, and $\int_t^\infty |c(u)|du \in L^1[0,\infty)$.

Acknowledgement . The author is very grateful to the referee for his useful comments.

References

[1] T. Burton, Uniform asymptotic stability in functional differential equations, *Proc. Amer. Math. Soc.*, **68** (1978), 195–199.

[2] T. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[3] T. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, Florida, 1985.

[4] T. Burton and L. Hatvani, Stability theorems for nonautonomous functional differential equations by Liapunov functionals, *Tôhoku Math. J.*, **41** (1989), 65–104.

[5] T. Burton and S. Zhang, Unified boundedness, periodicity and stability in ordinary and functional differential equations, *Ann. Math. Pura. Appl.*, **145** (1986), 129–158.

[6] R. Driver, Existence and stability of solutions of a delay-differential system, *Archiv. Rat. Mech. Anal.*, **10** (1962), 401–426.

[7] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[8] L. Hatvani, On the asymptotic stability of the solution of functional differential equations, *Coll. Math. Soc. J. Bolyai*, **53** (1988), 227–238.

[9] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities, Theory and Applications*, Academic Press, New York - London, 1969.

[10] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan, Tokyo, 1966.

[11] Wu Jianhong and L. Hatvani, On the boundedness of solutions of nonautonomous differential equations, *Acta Sci. Math. (Szeged)*, **53** (1989), 267–285.

Y. Ko, Departement of Mathematics Education, Cheju National University, Cheju, Korea 690-756