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COMPARISONS OF TWO NEW OPERATORS AND OTHERS IN FUZZY MATRICES

Kyung-Tae Kang and Eun-Sook Bang
Department of Mathematics, Cheju National University,
Jeju 690-756, Republic of Korea
e-mail : kangkt@cheju.ac.kr and banges@cheju.ac.kr

Abstract

There have been many researches on fuzzy relational equations. The fuzzy matrices are used when fuzzy uncertainty occurs in a problem. In this paper, under the general framework of fuzzy matrix theory, we introduce two new binary fuzzy operators $\sqcup$ and $\sqcap$. Some properties on $\sqcup$ and $\sqcap$ are presented in this paper. Also, we give some comparisons of these new operators and existing operators $\vee$, $\wedge$ $\oplus$, $\odot$ and $+_\lambda$.

Keywords: Fuzzy matrix, fuzzy operators, pre-vanishing operators.

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1 Introduction and Definitions

Fuzzy matrices play major rule in various areas such as mathematics, physics, statistics, engineering, social sciences and many others. Now a days probability, fuzzy sets, intuitionistic fuzzy sets, vague sets and rough sets are used as mathematical tools for dealing uncertainties. Fuzzy matrices arise in many applications, and several authors ([1]-[10]) presented a number of results on fuzzy matrices.

Let $\mathbf{F} = [0,1]$ be the real closed interval. We define some operators on fuzzy matrices whose elements are confined in $\mathbf{F}$. For all $x, y, \lambda \in \mathbf{F}$, the following operators are defined:

(i) $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$,

(ii) $x \oplus y = x + y - x \cdot y$ and $x \odot y = x \cdot y$,

(iii) $x^c = 1 - x$,

(iv) $x +_\lambda y = \lambda x + (1 - \lambda) y$.

Now, we define two new operators $\sqcup$ and $\sqcap$ as follows: for all $x, y \in \mathbf{F}$,

(v) $x \sqcup y = \begin{cases} 1 & \text{if } x > y \\ y & \text{if } x \leq y \end{cases}$ and $x \sqcap y = \begin{cases} y & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases}$

In fact, $x \sqcup y$ was introduced in [1]. We remark that for all $x, y \in \mathbf{F}$, $x \sqcup y, x \sqcap y \in \{0, 1, y\}$. For this reason, we call these two operators $\sqcup$ and $\sqcap$ *pre-vanishing operators.*

We may be noted that the values of $x \vee y$, $x \wedge y$, $x \oplus y$, $x \odot y$, x^c, $x +_\lambda y$, $x \sqcup y$ and $x \sqcap y$ belong to $\mathbf{F}$.

Let $\mathcal{M}_n(\mathbf{F})$ denote the set of all $n \times n$ matrices with entries in $\mathbf{F}$. The matrices I_n and O_n are the $n \times n$ identity matrix and zero matrix, respectively.

For all $A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}_n(\mathbf{F})$ and for all $\lambda \in \mathbf{F}$, the following operators are defined:

(i) $A \vee B = [a_{ij} \vee b_{ij}]$ and $A \wedge B = [a_{ij} \wedge b_{ij}]$,

(ii) $A \oplus B = [a_{ij} \oplus b_{ij}]$ and $A \odot B = [a_{ij} \cdot b_{ij}]$,

(iii) $A^c = [1 - a_{ij}]$,

(iv) $A +_\lambda B = [a_{ij} +_\lambda b_{ij}]$,

(v) $A \sqcup B = [a_{ij} \sqcup b_{ij}]$ and $A \sqcap B = [a_{ij} \sqcap b_{ij}]$,

(vi) $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all i and j.

Throughout this paper, we assume that $\lambda \in \mathbf{F}$ so that $0 \le \lambda \le 1$. Furthermore we assume that $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ and $D = [d_{ij}]$ are fuzzy matrices.

In [7], Shyamal and Pal characterized some properties of operators $\oplus$ and $\odot$ with pre-defined operators.

In this paper, we introduce new binary operators $\sqcup$ and $\sqcap$ on fuzzy matrices. Also, some properties of the fuzzy matrices over these new operators and some pre-defined operators are presented.

2 Some comparisons

We note that operators $\vee$, $\wedge$, $\oplus$ and $\odot$ are commutative, while $+_\lambda$, $\sqcup$ and $\sqcap$ are not. Furthermore, we can easily show that for all $x, y, z \in \mathbf{F}$

(2.1) $x \vee x = x \wedge x = x \sqcup x = x, \quad 0 \sqcup x = x \quad$ and $\quad 1 \sqcup x = x \sqcup 1 = 1,$

(2.2) $x \sqcap x = 0 \quad$ and $\quad x \sqcap 0 = 0 \sqcap x = 0,$

(2.3) $x \sqcap y = 0$ and $y \sqcap x = 0 \quad$ implies $\quad x = y,$

(2.4) $x \sqcap y \le y \quad$ and $\quad (x \sqcap y) \sqcap y = 0,$

(2.5) $(x \sqcap y) \sqcap (x \sqcap z) \le z \sqcap y$ if and only if $((x \sqcap y) \sqcap (x \sqcap z)) \sqcap (z \sqcap y) = 0.$

Property 1. *Let A and B be matrices in $\mathcal{M}_n(\mathbf{F})$. Then we have*

$$A \sqcap B \le A \wedge B \le A \vee B \le A \sqcup B.$$

Proof. Let x_{ij} and y_{ij} be $(i,j)^{\text{th}}$ entries of $A \sqcap B$ and $A \wedge B$, respectively. Then we have that $x_{ij} = \begin{cases} b_{ij} & \text{if } a_{ij} > b_{ij} \\ 0 & \text{if } a_{ij} \le b_{ij} \end{cases}$ and $y_{ij} = \min\{a_{ij}, b_{ij}\}$. Thus if $a_{ij} > b_{ij}$, then $x_{ij} = b_{ij} = y_{ij}$ and if $a_{ij} \le b_{ij}$, then $x_{ij} = 0 \le a_{ij} \le y_{ij}$. It follows that $A \sqcap B \le A \wedge B$. It is obvious that $A \wedge B \le A \vee B$ by the definitions of $\wedge$ and $\vee$.

Now, we remained to show that $A \vee B \le A \sqcup B$. Let z_{ij} and w_{ij} be $(i,j)^{\text{th}}$ entries of $A \vee B$ and $A \sqcup B$, respectively. Then $z_{ij} = \max\{a_{ij}, b_{ij}\}$ and $w_{ij} =$

$\begin{cases} 1 & \text{if } a_{ij} > b_{ij} \\ b_{ij} & \text{if } a_{ij} \le b_{ij} \end{cases}$, and hence $z_{ij} = a_{ij} \le 1 = w_{ij}$ and $z_{ij} = b_{ij} = w_{ij}$ according as $a_{ij} > b_{ij}$ or $a_{ij} \le b_{ij}$. Therefore we have that $A \vee B \le A \sqcup B$. ■

We note that De Morgan's laws (over complement) for two operators $*$ and $\circ$ are

$$(A * B)^c = A^c \circ B^c \quad \text{and} \quad (A \circ B)^c = A^c * B^c.$$

Property 2. *For matrices A and B in $\mathcal{M}_n(\mathbb{F})$, we have*

(i) $(A \vee B)^c = A^c \wedge B^c$,

(ii) $(A \wedge B)^c = A^c \vee B^c$,

(iii) $(A \oplus B)^c = A^c \odot B^c$,

(iv) $(A \odot B)^c = A^c \oplus B^c$,

Furthermore, if $a_{ij} \ne b_{ij}$ for all $i, j = 1, \ldots, n$, the following are satisfied:

(v) $(A \sqcup B)^c = A^c \sqcap B^c$,

(vi) $(A \sqcap B)^c = A^c \sqcup B^c$.

Proof. (i) Let x_{ij} and y_{ij} be the $(i, j)^{\text{th}}$ entries of $(A \vee B)^c$ and $A^c \wedge B^c$, respectively. Now, we will show that $x_{ij} = y_{ij}$ for all $i, j = 1, \ldots, n$. Note that

$$x_{ij} = (a_{ij} \vee b_{ij})^c \quad \text{and} \quad y_{ij} = a_{ij}^c \wedge b_{ij}^c.$$

If $a_{ij} \le b_{ij}$, then $1 - a_{ij} \ge 1 - b_{ij}$ (equivalently, $a_{ij}^c \ge b_{ij}^c$) and hence

$$x_{ij} = (a_{ij} \vee b_{ij})^c = b_{ij}^c = a_{ij}^c \wedge b_{ij}^c = y_{ij}.$$

Similarly, for the case of $a_{ij} > b_{ij}$, we have $x_{ij} = a_{ij}^c = y_{ij}$. Therefore $(A \vee B)^c = A^c \wedge B^c$.

(ii) Similar to (i).

(iii) Let x_{ij} and y_{ij} be the $(i, j)^{\text{th}}$ entries of $(A \oplus B)^c$ and $A^c \odot B^c$, respectively. Then we have

$$\begin{aligned}
x_{ij} &= (a_{ij} \oplus b_{ij})^c \\
&= 1 - (a_{ij} \oplus b_{ij}) \\
&= 1 - (a_{ij} + b_{ij} - a_{ij}b_{ij}) \\
&= 1 - a_{ij} - b_{ij} + a_{ij}b_{ij} \\
&= (1 - a_{ij})(1 - b_{ij}) \\
&= a_{ij}^c \cdot b_{ij}^c \\
&= y_{ij}.
\end{aligned}$$

This shows that $(A \oplus B)^c = A^c \odot B^c$.

(iv) Similar to (iii).

(v) Let x_{ij} and y_{ij} be the $(i,j)^{\text{th}}$ entries of $(A \sqcup B)^c$ and $A^c \sqcap B^c$, respectively. Then we have

$$x_{ij} = 1 - (a_{ij} \sqcup b_{ij}) \quad \text{and} \quad y_{ij} = (1 - a_{ij}) \sqcap (1 - b_{ij}).$$

It follows from $a_{ij} \neq b_{ij}$ that either $a_{ij} > b_{ij}$ or $a_{ij} < b_{ij}$. For the former($a_{ij} > b_{ij}$), we have $1 - a_{ij} < 1 - b_{ij}$, and hence $a_{ij} \sqcup b_{ij} = 1$ and $(1 - a_{ij}) \sqcap (1 - b_{ij}) = 0$. Thus $x_{ij} = 1 - 1 = 0 = y_{ij}$ in this case. For the latter($a_{ij} < b_{ij}$), the parallel argument shows that $x_{ij} = 1 - b_{ij} = y_{ij}$. Therefore $(A \sqcup B)^c = A^c \sqcap B^c$.

(vi) Similar to (v). ■

In Property 2, let $a_{ij} = b_{ij}$ for some i and j. Then the following Remark shows that neither (v) nor (vi) are satisfied.

Remark 1. Let A and B be matrices in $\mathcal{M}_n(\mathbf{F})$ with $a_{11} = b_{11} = 0.5$. Then we have

$$(a_{11} \sqcup b_{11})^c = 0.5 \neq 0 = a_{11}^c \sqcap b_{11}^c$$

and

$$(a_{11} \sqcap b_{11})^c = 1 \neq 0.5 = a_{11}^c \sqcup b_{11}^c.$$

It follows that in general

$$(A \sqcup B)^c \neq A^c \sqcap B^c \quad \text{and} \quad (A \sqcap B)^c \neq A^c \sqcup B^c. \qquad \blacksquare$$

De Morgan's laws (over transpose) for two operators $*$ and $\circ$ are

$$(A * B)^T = A^T \circ B^T \quad \text{and} \quad (A \circ B)^T = A^T * B^T,$$

where A^T is the transpose of A. Shyamal and Pal ([7]) showed that two operators $\oplus$ and $\odot$ does not satisfy the De Morgan's laws over transpose. For two operators $\sqcup$ and $\sqcap$, we obtain the same result as following:

Property 3. *Let A, B and C be matrices in $\mathcal{M}_n(\mathbf{F})$. Then*

(i) $(A \sqcup B)^T = A^T \sqcup B^T$ *and* $(A \sqcap B)^T = A^T \sqcap B^T$,

(ii) *if* $A \leq B$, *then* $A \sqcup C \leq B \sqcup C$ *and* $A \sqcap C \leq B \sqcap C$.

Proof. (i) Let x_{ij} and y_{ij} be $(i,j)^{\text{th}}$ entries of $A \sqcup B$ and $A^T \sqcup B^T$, respectively. Then $z_{ij} = x_{ji}$ is the $(i,j)^{\text{th}}$ entry of $(A \sqcup B)^T$. To show that $(A \sqcup B)^T = A^T \sqcup B^T$, we suffice to claim that $z_{ij} = y_{ij}$ for all $i,j = 1, \ldots, n$, equivalently $x_{ji} = y_{ij}$ for all $i,j = 1, \ldots, n$. Now, $x_{ji} = a_{ji} \sqcup b_{ji}$ and $y_{ij} =$ and

$$y_{ij} = \text{the } (i,j)^{\text{th}} \text{ entry of } A^T \sqcup \text{the } (i,j)^{\text{th}} \text{ entry of } B^T = a_{ji} \sqcup b_{ji} = x_{ji},$$

and hence $(A \sqcup B)^T = A^T \sqcup B^T$. By the similar argument, we also obtain that $(A \sqcap B)^T = A^T \sqcap B^T$.

(ii) Let x_{ij}, y_{ij}, z_{ij} and w_{ij} be $(i,j)^{\text{th}}$ entries of $A \sqcup C$, $B \sqcup C$, $A \sqcap C$ and $B \sqcap C$, respectively. Then we have

$$x_{ij} = \begin{cases} 1 & \text{if } a_{ij} > c_{ij} \\ c_{ij} & \text{if } a_{ij} \leq c_{ij} \end{cases} \quad \text{and} \quad y_{ij} = \begin{cases} 1 & \text{if } b_{ij} > c_{ij} \\ c_{ij} & \text{if } b_{ij} \leq c_{ij}, \end{cases}$$

$$z_{ij} = \begin{cases} c_{ij} & \text{if } a_{ij} > c_{ij} \\ 0 & \text{if } a_{ij} \leq c_{ij} \end{cases} \quad \text{and} \quad w_{ij} = \begin{cases} c_{ij} & \text{if } b_{ij} > c_{ij} \\ 0 & \text{if } b_{ij} \leq c_{ij}. \end{cases}$$

To show that $A \sqcup C \leq B \sqcup C$ and $A \sqcap C \leq B \sqcap C$, we suffice to claim that $x_{ij} = 1$ implies $y_{ij} = 1$, and $z_{ij} = c_{ij}$ implies $w_{ij} = c_{ij}$. Suppose that $x_{ij} = 1$ and $z_{ij} = c_{ij}$. Then we have $a_{ij} > c_{ij}$, and hence $b_{ij} > c_{ij}$ because $A \leq B$. Therefore we have $y_{ij} = 1$ and $w_{ij} = c_{ij}$. Hence $A \sqcup C \leq B \sqcup C$ and $A \sqcap C \leq B \sqcap C$. ■

Let A, B and C be matrices in $\mathcal{M}_n(\mathbf{F})$ with $A \leq B$. The following Remark shows that $C \sqcup A \nleq C \sqcup B$, $C \sqcup A \ngeq C \sqcup B$, $C \sqcap A \nleq C \sqcap B$ and $C \sqcap A \ngeq C \sqcap B$.

Remark 2. Let

$$A = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 & 0.5 \\ 0.5 & 0.6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.7 \end{bmatrix}$$

with $A \leq B$. Let $C \sqcup A = [x_{ij}]$, $C \sqcup B = [y_{ij}]$, $C \sqcap A = [z_{ij}]$ and $C \sqcap B = [w_{ij}]$. Then we have

$$x_{11} = 0.3 \sqcup 0.1 = 1 > 0.3 = 0.3 \sqcup 0.3 = y_{11},$$

$$x_{12} = 0.2 \sqcup 0.2 = 0.2 < 0.5 = 0.2 \sqcup 0.5 = y_{12},$$

$$z_{21} = 0.5 \sqcup 0.3 = 0.3 > 0 = 0.5 \sqcup 0.5 = w_{21}$$

and

$$z_{22} = 0.7 \sqcup 0.4 = 0.4 < 0.6 = 0.7 \sqcup 0.6 = w_{22}.$$

It follows that

$$C \sqcup A \nleq C \sqcup B,$$

$$C \sqcup A \ngeq C \sqcup B,$$

$$C \sqcap A \nleq C \sqcap B$$

and

$$C \sqcap A \ngeq C \sqcap B.$$ ■

Notice that the operators $\oplus$ and $\odot$ are commutative as well as associative. Therefore the following are obvious:

$$A * B = B * A \quad \text{and} \quad (A * B) * C = A * (B * C), \tag{2.6}$$

where $*$ is either $\oplus$ or $\odot$. But the operators $\sqcup$ and $\sqcap$ are neither commutative nor associative. The following Remark shows that (2.6) may be not true for $\sqcup$ and $\sqcap$.

Remark 3. Let $A = \begin{bmatrix} 0.2 & 0.2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.1 & 0.3 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0.2 & 0.1 \\ 1 & 1 \end{bmatrix}$ be matrices in $\mathcal{M}_2(\mathbf{F})$. Then we have

$$A \sqcup B = \begin{bmatrix} 1 & 0.3 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0.2 & 0.1 \\ 1 & 1 \end{bmatrix} = B \sqcup A,$$

$$A \sqcap B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix} = B \sqcap A,$$

$$(A \sqcup B) \sqcup C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0.2 & 1 \\ 1 & 1 \end{bmatrix} = A \sqcup (B \sqcup C),$$

$$(A \sqcap B) \sqcap C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix} = A \sqcap (B \sqcap C).$$ ■

The following are some results of properties of operators $\oplus$ and $\odot$ with $\vee$ which are proved by Shyamal and Pal.

Property 4. ([7]) *Let A, B and C be matrices in $\mathcal{M}_n(\mathbf{F})$. Then*

(i) $A \odot B \leq A \oplus B$,

(ii) $A \leq A \oplus A$ *and* $A \odot A \leq A$,

(iii) $I_n \oplus (A \oplus A^T) = I_n \vee (A \oplus A^T)$,

(iv) $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$,

(v) $A \vee (B \oplus C) \leq (A \vee B) \oplus (A \vee C)$.

The analogue results of operators $\sqcup$ and $\sqcap$ with $\vee$ are satisfied as following:

Property 5. *Let A, B and C be matrices in $\mathcal{M}_n(\mathbf{F})$. Then*

(i) $A \sqcap B \leq A \sqcup B$,

(ii) $A = A \sqcup A$ *and* $A \sqcap A = O_n$,

(iii) $I_n \sqcup (A \sqcup A^T) = I_n \vee (A \sqcup A^T)$,

(iv) $A \sqcup (B \vee C) \leq (A \sqcup B) \vee (A \sqcup C)$.

Proof. (i) was proved in Proposition 1.

(ii) is obvious by properties (2.1) and (2.2).

(iii) Let x_{ij} and y_{ij} be $(i,j)^{\text{th}}$ entries of $I_n \sqcup (A \oplus A^T)$ and $I_n \vee (A \sqcup A^T)$, respectively. For the case of $i = j$, clearly $y_{ii} = 1$ and from (2.1), we have

$$x_{ii} = \begin{cases} 1 & \text{if } 1 > a_{ii} \\ a_{ii} & \text{if } 1 \leq a_{ii}. \end{cases}$$

Notice $1 \leq a_{ii}$ means that $1 = a_{ii}$, and hence $x_{ii} = 1$ so that $x_{ii} = y_{ii}$ for all $i = 1, \ldots, n$. For the case of $i \neq j$, clearly $y_{ij} = a_{ij} \sqcup a_{ji}$ and $x_{ij} = 0 \sqcup (a_{ij} \sqcup a_{ji}) = a_{ij} \sqcup a_{ji} = y_{ij}$. Therefore we have $x_{ij} = y_{ij}$ for all $i, j = 1, \ldots, n$ and hence $I_n \sqcup (A \sqcup A^T) = I_n \vee (A \sqcup A^T)$.

(iv) Let x_{ij} and y_{ij} be $(i,j)^{\text{th}}$ entries of $A \sqcup (B \vee C)$ and $(A \sqcup B) \vee (A \sqcup C)$, respectively. We will show that $x_{ij} \leq y_{ij}$ for all $i, j = 1, \ldots, n$. Now

$$x_{ij} = \begin{cases} 1 & \text{if } a_{ij} > \max\{b_{ij}, c_{ij}\} \\ \max\{b_{ij}, c_{ij}\} & \text{if } a_{ij} \leq \max\{b_{ij}, c_{ij}\} \end{cases}$$

and

$$y_{ij} = \max\{a_{ij} \sqcup b_{ij}, a_{ij} \sqcup c_{ij}\}.$$

Case 1) $a_{ij} > \max\{b_{ij}, c_{ij}\}$: Then $a_{ij} > b_{ij}$ and $a_{ij} > c_{ij}$, and hence $a_{ij} \sqcup b_{ij} = a_{ij} \sqcup c_{ij} = 1$ by the definition of $\sqcup$. Therefore, in this case, we have $x_{ij} = y_{ij} = 1$.

Case 2) $a_{ij} \leq \max\{b_{ij}, c_{ij}\}$: Then $x_{ij} = \max\{b_{ij}, c_{ij}\}$. If $a_{ij} \sqcup b_{ij} = 1$ or $a_{ij} \sqcup c_{ij} = 1$, then there is noting to prove $x_{ij} \leq y_{ij}$. Thus, we lose no generality to assuming that $a_{ij} \sqcup b_{ij} = b_{ij}$ and $a_{ij} \sqcup c_{ij} = c_{ij}$. That is, $y_{ij} = \max\{b_{ij}, c_{ij}\} = x_{ij}$. Thus, in this case, we also have $x_{ij} \leq y_{ij}$ for all $i, j = 1, \ldots, n$. ■

The following Remark shows that the equality of Property 5-(iv) may be not true. Furthermore the example shows that for operator $\vee$, Property 4-(v) may be false.

Remark 4. Consider three fuzzy matrices

$$A = \begin{bmatrix} 0.2 & 0.5 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0 \end{bmatrix}.$$

Let $A \sqcup (B \vee C) = [x_{ij}]$, $(A \sqcup B) \vee (A \sqcup C) = [y_{ij}]$, $A \vee (B \sqcup C) = [z_{ij}]$ and $(A \vee B) \sqcup (A \vee C) = [w_{ij}]$. By the calculations, we have

$$\begin{aligned} x_{11} &= 0.2 \sqcup (0.1 \vee 0.3) = 0.2 \sqcup 0.3 = 0.3, \\ y_{11} &= (0.2 \vee 0.1) \vee (0.2 \sqcup 0.3) = 1 \vee 0.3 = 1, \\ z_{12} &= 0.5 \vee (0.3 \sqcup 0.2) = 0.5 \vee 1 = 1, \\ w_{12} &= (0.5 \vee 0.3) \sqcup (0.5 \vee 0.2) = 0.5 \sqcup 0.5 = 0.5. \end{aligned}$$

Thus $x_{11} < y_{11}$ and $z_{12} > w_{12}$. ■

Let λ be given in $\mathbf{F}$. We remind that $x +_\lambda y = \lambda x + (1-\lambda)y$ for all $x, y \in \mathbf{F}$. Bang and Kang characterized relationships of operators $+_\lambda$, $\odot$, $\oplus$, $\wedge$ and $\vee$ as following:

Property 6. ([1]) *For matrices $A, B \in \mathcal{M}_n(\mathbf{F})$,*

$$A \odot B \leq A \wedge B \leq A +_\lambda B \leq A \vee B \leq A \oplus B.$$

Property 7. *Let A and B be matrices in $\mathcal{M}_n(\mathbf{F})$. Then we have*

$$A \sqcap B \leq A \wedge B \leq A +_\lambda B \leq A \vee B \leq A \sqcup B.$$

Proof. It follows from Property 6 that $A \wedge B \leq A +_\lambda B \leq A \vee B$. By Property 1, we have $A \sqcap B \leq A \wedge B$ and $A \vee B \leq A \sqcup B$. Thus the result follows. ■

3 Complement of fuzzy matrices

The complement of a fuzzy matrix is used to analysis the complement nature of any system. For example, if $A \in \mathcal{M}_n(\mathbf{F})$ represents the crowdness of a network at a particular time period, then its complement A^c represents the clearness at the same time period. Using the following results, we can study the complement nature of a system with the help of original fuzzy matrix.

The operator complement obey the De Morgan's laws for the operators $\oplus$ and $\odot$. This is established by Shyamal and Pal as following:

Property 8. ([7]) *For the fuzzy matrices A and B,*

(i) $(A \oplus B)^c = A^c \odot B^c$,

(ii) $(A \odot B)^c = A^c \oplus B^c$,

(iii) $(A \oplus B)^c \leq A^c \oplus B^c$,

(iv) $(A \odot B)^c \geq A^c \odot B^c$.

Bang and Kang obtained the following results:

Property 9. ([1]) *Let λ be given in $\mathbf{F}$. Then for the matrices A and B in $\mathcal{M}_n(\mathbf{F})$,*

(i) $(A +_\lambda B)^c = A^c +_\lambda B^c$,

(ii) $(A +_\lambda B)^c \geq A^c \odot B^c$,

(iii) $(A \odot B)^c \geq A^c +_\lambda B^c$,

(iv) $(A \sqcup B)^c \leq A^c \sqcup B^c$.

Now, we are interesting inequalities of complements of matrices for the operators $\sqcup$ and $\sqcap$.

Property 10. *For matrices A and B in $\mathcal{M}_n(\mathbf{F})$, we have*

(i) $A^c \sqcap B^c \leq (A \sqcap B)^c$,

(ii) $A^c \sqcap B^c \leq (A \sqcup B)^c \leq (A \sqcap B)^c \leq A^c \sqcup B^c$.

Proof. (i) Let x_{ij} and y_{ij} be the $(i,j)^{\text{th}}$ entries of $A^c \sqcap B^c$ and $(A \sqcap B)^c$, respectively. Then we have

$$x_{ij} = a^c_{ij} \sqcap b^c_{ij} = \begin{cases} b^c_{ij} & \text{if } a^c_{ij} > b^c_{ij} \\ 0 & \text{if } a^c_{ij} \leq b^c_{ij} \end{cases} = \begin{cases} 1 - b_{ij} & \text{if } a_{ij} < b_{ij} \\ 0 & \text{if } a_{ij} \geq b_{ij} \end{cases}$$

and

$$y_{ij} = (a_{ij} \sqcap b_{ij})^c = 1 - (a_{ij} \sqcap b_{ij}) = \begin{cases} 1 - b_{ij} & \text{if } a_{ij} > b_{ij} \\ 1 & \text{if } a_{ij} \leq b_{ij}. \end{cases}$$

Therefore we have $x_{ij} \leq y_{ij}$ for all $i, j = 1, \ldots, n$, equivalently $A^c \sqcap B^c \leq (A \sqcap B)^c$.

(ii) Let $A^c \sqcap B^c = [x_{ij}]$, $(A \sqcup B)^c = [y_{ij}]$, $(A \sqcap B)^c = [z_{ij}]$ and $A^c \sqcup B^c = [w_{ij}]$. Now, we will show that $x_{ij} \leq y_{ij} \leq z_{ij} \leq w_{ij}$ for all $i, j = 1, \ldots, n$. By the proof of (i), we have

$$\text{(3.1)} \qquad x_{ij} = \begin{cases} 1 - b_{ij} & \text{if } a_{ij} < b_{ij} \\ 0 & \text{if } a_{ij} \geq b_{ij} \end{cases} \quad \text{and} \quad z_{ij} = \begin{cases} 1 & \text{if } a_{ij} \leq b_{ij} \\ 1 - b_{ij} & \text{if } a_{ij} > b_{ij}. \end{cases}$$

Furthermore, we have

$$\text{(3.2)} \qquad y_{ij} = \begin{cases} 1 - b_{ij} & \text{if } a_{ij} \leq b_{ij} \\ 0 & \text{if } a_{ij} > b_{ij} \end{cases} \quad \text{and} \quad w_{ij} = \begin{cases} 1 & \text{if } a_{ij} < b_{ij} \\ 1 - b_{ij} & \text{if } a_{ij} \geq b_{ij}. \end{cases}$$

It follows from (3.1) and (3.2) that $x_{ij} \leq y_{ij} \leq z_{ij} \leq w_{ij}$ for all $i, j = 1, \ldots, n$. Therefore we conclude that $A^c \sqcap B^c \leq (A \sqcup B)^c \leq (A \sqcap B)^c \leq A^c \sqcup B^c$. ■

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