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약간 오목한 벽 위를 흐르는 층류 경계층 흐름에서 대류 불안정성의 발생

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The Onset of Convective Instabilities in the Laminar Boundary Layer Flow over the Slightly Concave Walls

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ABSTRACT

The onset of convective instability in the laminar boundary layer over the slightly curved walls is analyzed theoretically and compared with existing experimental data. A new set of stability equations are derived by using propagation theory based on linear stability theory and momentary instability concept. In this analysis the disturbances are assumed to have the form of a longitudinal vortices and also to experience the streamwise growth. It is found that upon the onset of instability, the disturbances do not grow exponentially in streamwise direction, and that the disturbances are mainly confined to the velocity boundary layer. The present study predicts the experimental results more reasonably.

Key Words : Gortler Vortex, Laminar Boundry Layer Flow, Propagation Theory

I. Introduction

It is well-known that in the primary laminar flows along concavely curved walls, the destabilizing action of the centrifugal forces can produce secondary motion in form of vortices. The related hydrodynamic instabilities usually lead to Taylor vortices in the flow between rotating concentric cylinders or Gortler ones in the boundary layer flow. Since this kind of secondary flow occurs in wide range of scientific and engineering fields such as the design of high-efficiency curved parallel plate heat exchanger, the cooling of turbine blades and in the design of the modern supercritical airfoils employing laminar flow control, many researchers have interests in the onset of secondary motion. In this classical problem, the roll-type convective motions, known as Gortler vortices occur when the Gortler number exceeds a critical value[1]. The Gortler number means the ratio of the centrifugal force and the viscous force. The basic mechanism for this vortex is identical to that

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shown by Rayleighfor rotating inviscid fluid and by Taylor for a rotating viscous fluid[2]. This instability motion is driven by centrifugal forces associated with the change in direction of motion forced on the fluid by the geometry of the boundary.

To analyze this problem, the local stability analysis where the streamwise growth of disturbances was neglected and parallel flow was assumed has been employed[3]. After Gortler, many attempts were made to correct, supplement and extend his theory. Employing the effect of vertical component of base flow, Floryan and Saric[4](1982) reformulated this problem in a coordinate system based on streamlines and potential lines and determined the critical Gotler number to be Go = 0.4638. Numerous theoretical and experimental studies were reviewed by Floyan and Saric[4], Floryan[5] and Schlichting and Gersten[6].

The purpose of this study is to examine the onset of convective instability of the laminar boundary layerflow over the slightly curved walls by employing propagation theory. Propagation theory is based on scaling and self-similar transformation under linear theory, which has been used with success in the stability analysis of Bnard-type convection[7] and in the onset of Taylor-like vortice in the time dependent Couette flows. We will extend propagation theory to convective instability of the laminar boundary layer flow over the slightly curved walls. For this specific system, the instability criteria obtained by propagation theory willbe compared with available experimental and previous theoretical results.

II. Base Flow Field

The system considered here is the laminar boundary layer flow over the over the slightly curved walls as shown in Fig. 1. The radius of curvature R is assumed to be much higher than boundary layer thickness Δ. A Newtonian fluid flows along the X-direction with free stream velocity U_∞. In this case, the base flow fields are governed by the following equations1);

$$\frac{R}{R+Z}\frac{\partial U_0}{\partial X}+\frac{\partial W_0}{\partial Z}+\frac{W_0}{R+Z}=0 \qquad (1)$$

$$\frac{R}{R+Z}U_0\frac{\partial U_0}{\partial X}W_0\frac{\partial U_0}{\partial Z}+\frac{U_0W_0}{R+Z}= -\frac{1}{\rho}\frac{R}{(R+Z)}\frac{\partial P_0}{\partial X} +\nu\left\{\nabla_1^2U_0+\frac{2R}{(R+Z)^2}\frac{\partial W_0}{\partial X}-\frac{U_0}{(R+Z)^2}\right\} \qquad (2)$$

$$\frac{R}{R+Z}U_0\frac{\partial W}{\partial X}W_0\frac{\partial W_0}{\partial Z}+\frac{U_0^2}{R+Z}=-\frac{1}{\rho}\frac{\partial P_0}{\partial Z} +\nu\left\{\nabla_1^2W_0+\frac{2R}{(R+Z)^2}\frac{\partial U_0}{\partial X}-\frac{W_0}{(R+Z)^2}\right\} \qquad (3)$$

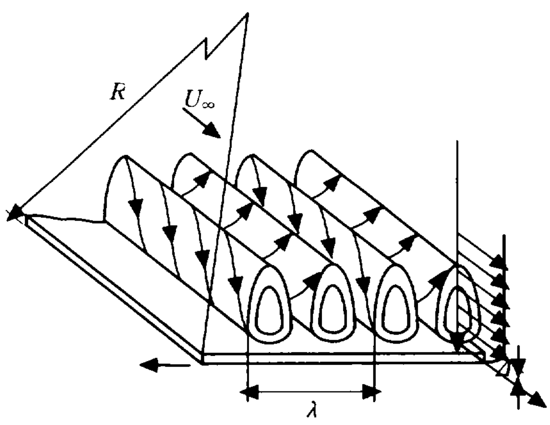

Fig. 1. Schematic diagram of system considered here.

where ∇_1^2 is a modified Laplacian operator. With the assumption of slightly curved wall ($R \approx 1/\Delta$) and the definition of stream function Ψ,

$$U=\frac{\partial\Psi}{\partial Z} \quad \text{and} \quad W=-\frac{\partial\Psi}{\partial X} \qquad (4)$$

the above equations can be approximated by well-known Blasius equation:

$$f'' + \frac{1}{2} ff' = 0 \tag{5}$$

with the following boundary conditions

$$f = f' = 0 \quad \text{at} \quad \zeta = 0 \tag{6.a}$$

$$f' = 1 \quad \text{at} \quad \zeta \to \infty \tag{6.b}$$

where the prime denotes the differentiation with respect to ζ. The dimensionless variables are defined by $f = \Psi(\nu U_\infty X)^{-1/2}$ and $\zeta = Z(U_\infty \nu)^{1/2} X^{-1/2}$. The experimental justification of these simplification for the slightly curved wall was given by Liepman1).

III. Disturbance Equations

In linear stability theory, the physical quantities, such as velocity and pressure are expressed as linear sumof basic quantities which are represented previously section and infinitesimal perturbation quantities in the following forms.

$$\begin{aligned} \vec{U} &= \vec{U}_0 + \vec{U}_1 \\ P &= P_0 + P_1 \end{aligned} \tag{7}$$

where $\vec{U} = \vec{i}U + \vec{j}V + \vec{k}W$ denotes the velocity vector and P the pressure, respectively. The subscripts "0" and "1" represent basic and perturbation quantities, respectively. Invoking the linear stability theory, the following disturbance equations can be derived under the assumption of slightly curved wall5)

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} + \frac{\partial W_1}{\partial Z} = 0 \tag{8}$$

$$\left(\frac{\partial U_1}{\partial t} + U_0 \frac{\partial U_1}{\partial X} + U_1 \frac{\partial U_0}{\partial X} + W_0 \frac{\partial U_1}{\partial Z} + W_1 \frac{\partial U_0}{\partial Z}\right)$$

$$= -\frac{1}{\rho} \frac{\partial P}{\partial X} + \nu \nabla^2 U_1 \tag{9}$$

$$\left(\frac{\partial V_1}{\partial t} + U_0 \frac{\partial V_1}{\partial X} + W_0 \frac{\partial V_1}{\partial Z}\right)$$

$$= -\frac{1}{\rho} \frac{\partial P}{\partial Y} + \nu \nabla^2 V_1 \tag{10}$$

$$\left(\frac{\partial W_1}{\partial t} + U_0 \frac{\partial W_1}{\partial X} + W_1 \frac{\partial U_0}{\partial X} + W_0 \frac{\partial W_1}{\partial Z} + W_1 \frac{\partial W_0}{\partial Z}\right) = -\frac{\partial P}{\partial Z} + \nu \nabla^2 W_1 - \frac{2}{R} U_0 U_1 \tag{11}$$

Since the boundary layer thickness is proper length scale for the boundary layer flow, the disturbance equations of Eqs. (8)-(11) must be changed for this length scale. Therefore, the nondimensionalized variables are introduced as follows ;

$$\begin{aligned} &\tau = tU_\infty / L \\ &(x, y, z) = (X, Y Re_L^{1/2}, Z Re_L^{1/2})/L \\ &(u, v, w) = (G_L U_1, V_1 Re_L^{1/2}, W_1 Re_L^{1/2})/U_\infty \\ &(U, W) = (U_0, W_0 Re_L^{1/2})/U_\infty \\ &p = P_1/(\rho U_\infty^2 Re_L^{-1}) \end{aligned} \tag{12}$$

where $G_L = \frac{L}{R}\left(\frac{U_\infty L}{\nu}\right)^2$. It should be noted that streamwise velocity disturbance is nondimensionalized by U_∞ / G_L rather than U_∞. After this nondimensionalization process, the dimensionless equations governing the disturbances can be written as

$$\frac{1}{G_L} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{13}$$

$$\left(\frac{\partial u}{\partial \tau} + U \frac{\partial u}{\partial x} + u \frac{\partial U}{\partial x} + W \frac{\partial u}{\partial z} + G_L w \frac{\partial U}{\partial z}\right)$$

$$= -\frac{G_L}{Re_L} \frac{\partial p}{\partial x} + \left(\frac{1}{Re_L} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{14}$$

$$\left(\frac{\partial v}{\partial \tau}+U\frac{\partial v}{\partial x}+W\frac{\partial v}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial y}+\left(\frac{1}{Re_L}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \quad (15)$$

$$\left(\frac{\partial w}{\partial \tau}+U\frac{\partial w}{\partial x}+\frac{u}{G_l}\frac{\partial W}{\partial x}+W\frac{\partial w}{\partial z}+w\frac{\partial W}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\left(\frac{1}{Re_L}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad (16)$$

with the proper boundary conditions

$$u=v=w=0 \text{ at } z=0 \text{ and } z\to\infty \quad (17)$$

the terms involving $1/Re_L$, $\partial p/\partial x$, $\partial^2 u/\partial x^2$, $x^2\partial^2 v/\partial$ and $\partial^2 w/\partial x^2$ are neglected by involving the boundary layer assumption $(Re_L\to\infty)$[5]. In addition, we may write $U=\phi'$ and $W=-x^{-1/2}\frac{1}{2}\phi-\frac{1}{2}\zeta\phi'$.

IV. Propagation Theory

To examine the stability characteristics of this problem, we must find the minimum value of G_L that satisfies Eqs. (13)-(17) for a given x. The propagation theory employed to find the critical streamwise position X_c to mark the onset of convection is based on the assumption that disturbances are propagated mainly within the velocity boundary layer thickness Δ at $X_c \gg \Delta$. In this case the following scale analysis at $X\approx X_c$ would be valid for dimensional perturbed quantities of Eqs. (8), (9) and (11).

$$\nu\frac{U_1}{\Delta^2} \quad W_1\frac{U_0}{\Delta} \quad (18)$$

$$\nu\frac{W_1}{\Delta^2} \quad \frac{U_0U_1}{R} \quad (19)$$

The above relations are nondimensionalized as

$$u \quad \frac{w}{\delta^2} \quad (20)$$

$$G_L w\frac{\partial U}{\partial z} \quad \frac{u}{\delta^2} \quad (21)$$

where δ is the dimensionless boundary layer thickness, $\delta=\Delta/L=(x^{1/2}Re_L^{-1/2})$. This means that Gortler vortex occurs due to u and this incipient secondary flow is very weak at $x=x_c$. Because U has the magnitude of order of 1, G^* $G_L\delta^3$ is a constant for $\delta\ll 1$ from the above relations. In this viewpoint the streamwise base velocity and its perturbation have been nondimensionalized having different scales. Since δ is small in the region considered here, the relation of $|u|\gg|w|$ is kept but $|1/G_L\partial u/\partial x|$ has the same order of magnitude as $|\partial w/\partial z|$. The above relations are consistent with Eqs. (13)-(17). Now, we assume that steady disturbances are periodic with the wavenumber in the spanwise y-direction. From this and the continuity equation of Eq. (13), the following relation is obtained:

$$a\cdot v \quad \frac{w}{\delta} \quad (22)$$

where "a" represent the dimensionless wave number, which means spanwise periodicity of disturbance quantities. Based on the above relations, the relations of $u=\delta^n u^*$, $v=(\delta^{n+1}/a)v^*$ and $w=\delta^2 w^*$ can be obtained.

Shen[8] suggested the momentary instability condition: the temporal growth rate of the perturbation quantities (r_1) should exceed that of the base flow (r_0). By generalizing this momentary instability conception into relative instability concept for the present system, the marginal condition can be determined at the position where $r_0=r_1$. In the present system the dimensionless streamwise growth rates are defined as quantities of streamwise components:

$$r_0=\frac{1}{\langle U\rangle}\frac{d\langle U\rangle}{dx} \tag{23}$$

$$r_1=\frac{1}{\langle u\rangle}\frac{d\langle u\rangle}{dx} \tag{24}$$

where $\langle quantity\rangle=\left[\left(\int_A (quantity)^2 dA\right)/A\right]^{1/2}$ and $dA=\lambda dz$. Here A denotes the cross sectional area of one vortex roll pair in $x-y$ plane. From the base velocity distribution given by Eqs. (4)~(6), r_0 can be obtained as:

$$r_0=\frac{1}{4x} \tag{25}$$

For the case of $n=0$, the condition of $r_0=r_1$ is fulfilled at $x=x_c$, which will be discussed later. If the related process is still laminar-diffusional flow dominant with $G^*=constant$ at $x=x_c$, it is probable that $u(x,z)=u^*(\zeta)$. This means that the amplitude function of streamwise velocity disturbances follows the behavior of U. So, for the most dangerous longitudinal vortex rolls the disturbance quantities are expressed as

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix}=\begin{bmatrix} u^*(\zeta) \\ \delta v^*(\zeta)/a \\ \delta^2 w^*(\zeta) \\ \delta p^*(\zeta) \end{bmatrix}\exp(iay) \tag{26}$$

Substituting Eq. (26) into Eqs. (13) to (17), and eliminating v^* and p^* we obtain the following set of stability equation

$$(D^2-a^{*2})u^{**}=U\left(-\frac{1}{2}\zeta Du^{**}\right)+u^{**}\left(-\frac{1}{2}\zeta DU\right)+W^*Du^{**}+w^*DU \tag{27}$$

$$\begin{aligned}(D^2-a^{*2})w^*&=-2a^{*2}G^*Uu^{**}+\frac{1}{2}\zeta D^4u^{**}\\&+a^{*2}Du^{**}-\frac{1}{2}\zeta a^{*2}D^2u^{**}\\&+\frac{1}{2}DU\left(Dw^*-\frac{1}{2}\zeta Du^{**}\right)\\&+U\left(D^2w^*-\frac{1}{2}Du^{**}-\frac{1}{2}\zeta D^2u^{**}\right)\\&-\frac{1}{2}\zeta DU\left(D^2w^*-\frac{1}{2}Du^{**}-\frac{1}{2}\zeta D^2u^{**}\right)\\&-\frac{1}{2}\zeta U\left(D^3w^*-\frac{1}{2}D^2u^{**}-\frac{1}{2}\zeta D^3u^{**}\right)\\&+DW^*\left(D^2w^*-\frac{1}{2}Du^{**}-\frac{1}{2}\zeta D^2u\right)\\&+W^*\left(D^3w^*-D^2u^{**}-\frac{1}{2}\zeta D^3u\right)\\&-U\left(a^{*2}w^*-\frac{1}{2}\zeta a^{*2}Dw^*\right)\\&+a^{*2}u^{**}\left(\frac{1}{2}W^*+\frac{1}{2}\zeta DW^*\right)\\&-W^*a^{*2}Dw^*-a^{*2}w^*DW^*\end{aligned} \tag{28}$$

with

$$u^{**}=w^*=Dw^*=0 \text{ at } \zeta=0 \text{ and } \infty \tag{29}$$

where $u^{**}=u^*/G^*$, $a^*=a\delta$, $G^*=G_L\delta^3$ and $W^*=\left\{\frac{1}{2}\phi-\frac{1}{2}\zeta\phi'\right\}$. The parameter a^* and G^* based on the length scaling factor of velocity boundary layer thickness are assumed to be eigenvalues. Now the principle of exchange of stability is employed, and the minimum value of G^* is sought. This whole procedure is the essence of the propagation theory.

The conventional local stability analysis with parallerl flow model neglects the terms involving $\partial(\cdot)/\partial x$ and set $W^*=0$ in Eqs. (13)-(16) in amplitude coordinates x and z. This results in $(D^2-a^{*2})u^*=G^*W^*DU$ and $(D^2-a^{*2})w^*=-2a^{*2}Uu^*$ instead of Eqs. (27) and (28)[9].

V. Solution Method

In order to solve the stability equation of Eq. (27)-(29)the basic flow field solution must be obtained, a priori. For this purpose the fourth or fifthorder Runge-Kutta-Fehlberg method is employed. The stability equations are solved by employing the outward shooting scheme. In order to integrate these stability equations the proper value of Du^*, D^2w^* and D^3w^* at $\zeta=0$ are assumed for a given a^*, since the stability equations and the boundary conditions are all homogeneous, D^2w^* at $\zeta=0$ can be assigned arbitrarily and the parameter G^* is assumed. This procedure can be understood easily by taking into account of properties of eigenvalue problems. Since all the initial conditions are provided, this initial value problem can be proceeded numerically.

Integration is performed from the heated surface $\zeta=0$ to a fictitious outer boundary with the fourth order Runge-Kutta-Gill method. If the guessed values of G^*, $Du^*(0)$ and $D^3w^*(0)$are correct, u^*, w^* and Dw^*will vanish at the outer boundary. To improve the initial guess the Newton-Raphson iteration is used. When convergence is achieved, the outer boundary is increased by predetermined value and the above procedure is repeated. Since the disturbances decay exponentially outside the thermal boundary layer, the incremental change in G^*also decays fast with an increase in outer boundary depth. This behavior enable us to extrapolate the eigenvalue G^* to the infinite depth. Using the similar procedure, the results from the local stability analysis are obtained.

VI. Results and Discussions

The predicted values based on the above numerical scheme constitute the stability curve, as shown in Fig. 2. From this figure the stability criteria of the minimum G^* is found to be 12.95 with its corresponding a^* value of 0.54. The eigenvalues G^* and a^* have the following forms;

$$G^* = Go^2 = \left(\frac{\Delta}{R}\right)\left(\frac{U_\infty \Delta}{\nu}\right)^2 \text{ and } a^* = \frac{2\pi\Delta}{\lambda} \quad (30)$$

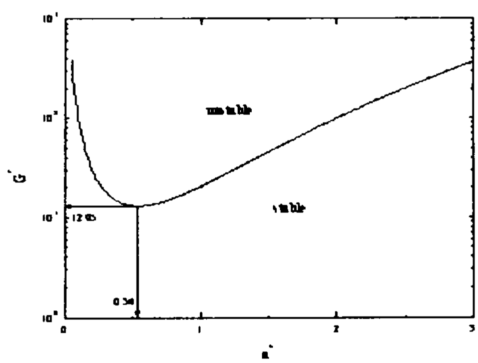

Fig. 2. Neutral stability curve.

With the relation of $\Delta_2/\Delta = 0.664$, where Δ_2 is the momentum thickness, the above critical conditions are given as

$$\left(\frac{\Delta_2}{R}\right)^{1/2} Re_{\Delta_2} = 1.95 \text{ and}$$

$$\Lambda = Re_\lambda \left(\frac{\lambda}{R}\right)^{1/2} = 141.9 \quad (31)$$

Now, the above results are compared with the available experimental and also some available predictions. Gortler9) employed the local stability theory with parallel flow assumption where the terms involving $\partial(\cdot)/\partial x$ were neglected and W^* was set to 0. His theoretical results were summarized as

$$\left(\frac{\Delta_2}{R}\right)^{1/2} Re_{\Delta_2} = 0.58 \quad \text{and}$$

$$\Lambda = Re_{\lambda}\left(\frac{\lambda}{R}\right)^{1/2} = 157.5 \quad (32)$$

Floryan and Saric4) reformulated this problem in a coordinate system based on streamlines and potential lines under the local stability assumption where the streamwise amplification was neglected. Their stability criteria were $Go = 0.4638$ and $a^{*} \to 0$ and can be transformed as

$$\left(\frac{\Delta_2}{R}\right)^{1/2} Re_{\Delta_2} = 0.25 \text{ and } \Lambda = Re_{\lambda}\left(\frac{\lambda}{R}\right)^{1/2} = \infty \quad (33)$$

In Fig. 3 the above predictions are also compared with experimental results. As shown in this figure, theoretical results are lower than experimental results. However, the present result predicts more reasonably than the previous one.

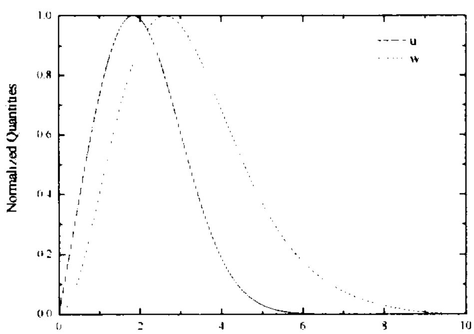

Fig. 3. Amplitude profiles at $x = x_c$

Since linear theory deals only with the growth of infinitesimal disturbances it is possible that the discrepancy between theory and experiment is due to the fact that only finite disturbances are actually observed. The infinitesimal disturbances must grow appreciably before they are observed.

At the critical conditions illustrated above, the amplitude functions of u^{*} and w^{*} are featured in Fig. 4, wherein the quantities have been normalized by the corresponding maximum magnitude u^{*}_{max} and w^{*}_{max}. It is seen that incipient streamwise velocity disturbances are confined mainly within the boundary layer thickness but vertical velocity disturbances are driven moreupward over the boundary layer thickness. Based on the distribution of incipient streamwise velocity disturbances, their streamwise growth rate can be obtained from Eq. (19): $r_0 = r_1 = 1/(4x_c)$. This means that for large G_L the growth rate at $x = x_c$ is inversely proportional to x_c and the present study bounds Shen's relative instability concept.

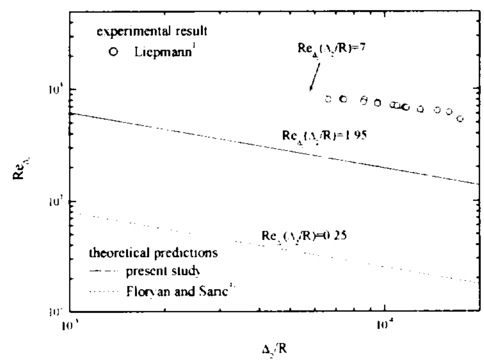

Fig. 4. Comparison of predictions with experimental data.

VII. Conclusions

The onset of convective instability of the laminar boundary layer flow over the slightly curved walls was analyzed. New stability equations were derived by employing propagation theory based on the linear stability analysis, under the assumption that the amplitudes of disturbances experience streamwise variation. The governing parameter was found to be $\left(\frac{\Delta}{R}\right)^{1/2} Re_{\Delta}$ or

$\left(\frac{\Delta_2}{R}\right)^{1/2} Re_{\Delta_2}$. Although the present study under-predicts the critical value of $\left(\frac{\Delta_2}{R}\right)^{1/2} Re_{\Delta_2}$ better agreement with experimental data was obtained by taking into account the streamwise variations of disturbances than by neglecting them. It is interesting that the velocity disturbances were confined mainly within the velocity boundary layer thickness.

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