On some Properties of the Upper Integral in Daniell Integration

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다니엘 積分에서의 上積分의 諸 性質

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Summary

The purpose of the present paper is to find that the upper integral in Daniell integration having the following properties; $L^* = \{f | \bar{I}(|f|) < \infty\}$ is complete and $\bar{I}$ is continuous. Moreover, monotone convergence theorem and Fatou's lemma in L^* hold.

1. Introduction

The approach to the Lebesgue-type integration is a base on the theory of measure. In some parts of analysis, the attention is focused much more on integrals than on measure. In the meantime, a novel approach to integration theory had been suggested in 1918 by P.J. Daniell. In brief, his idea was to treat an integral as a type of linear functional. Daniell's works was largely negelected for some twenty years, at the end of which period interest was revived by Bourbaki, presumable when plans were being laid for the relevant section of his future work in this field.

2. Preliminary Results

In the present section, for our further discussions, results obtained in our previous paper Ryu K.S. August, 1979 will be introduced without proof.

The following notations will be used throughout the present paper:

X is a compact Hausdorff topological space;

L is the set of all real-valued continuous functions on X;

I is a positive linear functional on L;

L_u is the set of all extended real-valued functions on X each of which is a limit of a monotone increasing sequence of functions in L;

For an arbitrary function f in L_u we define $I_u(f)$ by setting $I_u(f) = \lim I(f_n)$ where f_n in L and $f \uparrow f_n$;

For an arbitrary fuction f on X we define the upper integral $\bar{I}$ by setting $\bar{I}(f) = \inf\{I_u(g) \mid g \geq f$ and g in $L_u\}$. We define the lower integral $\underline{I}(f) = -\bar{I}(-f)$;

L_1 is the set of all extended real-valued functions on X each of which satisfied $\bar{I}(f) = \underline{I}(f)$ and $\bar{I}(f) < \infty$;

L^* is the set of all extended real-valued functions on X each of which satisfied $\bar{I}(|f|) < \infty$.

Using the above concepts, we obtain the following properties.

Proposition 2.1.

(1) If f is in L, then $I_u(f) = I(f)$.

(2) If a, b be nonnegative real numbers and let f, g be in L_u, $I_u(af + bg) = aI_u(f) + bI_u(g)$.

(3) $I_u(\underline{O}) = O$.

(4) If f and g are in L_u and $\underline{O} \leq f \leq g$, then $O \leq I_u(f) \leq I_u(g)$.

Proposition 2.2 For all f,g in R^X, the following statements hold.

(1) $\underline{I}(f) \leq \bar{I}(f)$.

(2) $\bar{I}(f+g) \leq \bar{I}(f) + \bar{I}(g)$.

(3) If $\underline{O} \leq f \leq g$ then $O \leq \bar{I}(f) \leq \bar{I}(g)$.

(4) If r is a nonnegative real number then $\bar{I}(rf) = r\bar{I}(f)$.

Proposition 2.3. L_u is a lattice.

Proposition 2.4. L^* is a linear-lattice.

Proposition 2.5. L_1 is a subset of L^*.

Proposition 2.6. For all f in L^*, $|\bar{I}(f)| \leq \bar{I}(|f|)$.

3. Continuity and Completeness

The purpose of this section is to find that $\bar{I}$ is continuous and that L^* is complete. We begin by establishing some lemmas.

Lemma 3.1. For all f and g in L^*, $n(|f|) = \bar{I}(f)$ is a seminorm on L^*.

Proof) For any f and g in L^*, we have

$n(f+g) = \bar{I}(|f+g|) \leq \bar{I}(|f|) + \bar{I}(|g|) = n(f) + n(g)$.

And for any f in L^* and any real number r, we have

$n(rf) = \bar{I}(|rf|) = |r|\,\bar{I}(|f|) = |r|\,n(f)$.

Hence, this lemma holds.

From the above lemma we know that L^* is a semimetric linear lattice with semimetric $d(f,g) = n(f-g)$.

Lemma 3.2. Let f and g be elements of L^*. Then $|\bar{I}(f) - \bar{I}(g)| \leq \bar{I}(|f-g|)$.

Consider the semimetric topology (L^*, d) and usual topology $(R,\mathcal{U})$. Using the lemma 3.2, we have

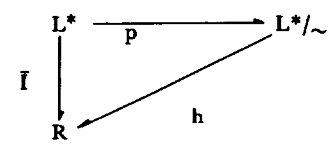
Theorem 3.3. The upper integral $\bar{I}:(L^*, d) \to (R,\mathcal{U})$ is continuous.

Proof) For arbitrary positive real number ϵ, there exists $\epsilon > 0$ such that $d(f,g) = n(f-g) = \bar{I}(|f-g|) < \epsilon$ implies

$|\bar{I}(f) - \bar{I}(g)| \leq \bar{I}(f-g) < \epsilon$. Hence, $\bar{I}$ is continuous.

Let us write $f \sim g$ if and only if $d(f,g) = 0$. It is clear that this is an equivalence relation in L^* which partitions L^* into equivalence classes, each class consists of all functions which are equivalent to a given one. If F and G are two equivalence classes, choose f in F and g in G, and define $d_e(F,G) = d(f,g)$, then we obtain an quotient space $L^*/\sim$ and $(L^*/\sim, d_e)$ is a metric space. Hence we easily have the following corollary.

Corollary 3.4 Let the projection $p: L^* \to L^*/\sim$ be an identification. Then there exists a continuous function h such that the diagram



is commutative.

Furthermore, we have $\bar{I}(|f|) = d(f,\underline{O}) = d_e(F,\underline{O}) = h(|p(f)|)$. Lastly, we show that

Theorem 3.5 (L^*,d) is complete.

Proof) Let (f_n) be a Cauchy sequence in L^*. We pick a subsequence (f_{m_i}) from it such that $d(f_{m_i}, f_{m_{i+1}}) < 2^{-i}$. Putting $f = f_{m_i} + (f_{m_{i+1}} - f_{m_i}) + \cdots\cdots$ Then $\bar{I}(|f|) \leq \bar{I}(|f_{m_i}|) + \bar{I}(|f_{m_{i+1}} - f_{m_i}|) + \cdots\cdots = \bar{I}(|f_{m_i}|) + 1 < \infty$.

Hence $f \in L^*$ which implies that (L^*,d) is complete.

By the above theorem, we directly have the following corollary.

Corrollary 3.6 $(L^*/\sim, d_e)$ is complete, that is, $(L^*/\sim, d_e)$ is a Branch space.

4. Monotone Convergence Theoream and Fatou's Lemma in L^*

One of the most fundamental results concerning Lebesgue-type integrals with respect to σ-additive measures now appears in the following disguise.

Theorem 4.1. Suppose f is the limit of a monotone sequence (f_n) of functions in L^*. Then $\bar{I}(f) = \lim \bar{I}(f_n)$

Proof) We easily obtain that $I(f) \geq I(f_n)$ for each n. Hence $\bar{I}(f) \geq \lim \bar{I}(f_n)$.

In proving the reverse one may assume that $\bar{I}(f_n) < \infty$ for each n.

It will be shown that for any $\epsilon > 0$ one may choose an increasing sequence g_n from L_u such that $f_n < g_n$ and $I_u(g_n) < \bar{I}(f_n) + \epsilon$.

If this be done, $g = \lim g_n$ will belong to L_u with $f \leq g$ and will show that $I_u(g) = \lim I_u(g_n) \leq \bar{I}(f_n) + \epsilon$.

Accordingly $\bar{I}(f) \leq I_u(g) \leq \lim \bar{I}(f_n) + \epsilon$.

Since ϵ is arbitrarily small, the proof will be complete.

To contruct the g_n one begins by choosing h_n in L_u with $h_n \geq f_n$ and such that $\bar{I}(f_n) \leq I_u(h_n) \leq \bar{I}(f_n) + \epsilon/2^n$, and then shows that the $g_n = \sup_{1<m<n} h_m$ satisfy the demands.

Evidently, $g_n \in L_u$ and $g_n \geq f_n$. The final step is to prove by induction on n that $I_u(g_n) \leq \bar{I}(f_n) + \epsilon(1 - 2^{-n})$.

Now this is true for $n = 1$.

Assume it true for $n = k$. One has $g_{k+1} = \sup_{1<m<k+1} h_m = \sup(g_k, h_{k+1})$

Since $g_k \geq f_k$ and $h_{k+1} \geq f_{k+1} \geq f_k$, so $\inf(g_k, h_{k+1}) \geq f_k$.

Also $\inf(g_k, h_{k+1}) + \sup(g_k, h_{k+1}) = g_k + h_{k+1}$.

Using the **proposition 2.1** and **proposition 2.3**, we have

$$\bar{I}(f_k) + I_u(g_{k+1}) \leq I_u(\inf(g_k, h_{k+1})) + I_u(\mathrm{Sup}(g_k, h_{k+1})) = \bar{I}(\inf(g_k, h_{k+1}) + \sup(g_k, h_{k+1})) =$$

$$\bar{I}(g_k + h_{k+1}) = I_u(g_k + h_{k+1}) = I_u(g_k) + I_u(h_{k+1}).$$

So, by inductive hypothesis,

$$I_u(g_{k+1}) \leq I_u(g_k) + I_u(h_{k+1}) - \bar{I}(f_k) \leq \bar{I}_u(f_k) + \epsilon\,(1 - 1/2^k) + \bar{I}(f_{k+1}) + \epsilon/2^{k+1} - \bar{I}(f_k) = \bar{I}(f_{k+1}) + \epsilon\,(1 - 1/2^{k+1}),$$ that is,

$I_u(g_n) \leq \bar{I}(f_n) + \epsilon\,(1 - 1/2^n)$ holds.

The proof ends by appeal to the mathematical induction.

By the above theorem, we obtain the fatou's lemma.

Corollary 4.2. Let (f_n) be an arbitrary sequence in L^*. Then $\bar{I}(\liminf f_n) \leq \liminf \bar{I}(f_n)$.

Proof) Consider the function $F_m = \inf_{n>m} f_n$. Then $\lim \inf_{n>m} f_n$ is that limit of a monotone sequence (F_m). Hence, by the above theorem,

$\bar{I}(\lim \inf_{n>m} f_n) = \lim \bar{I}(\inf_{n>m} f_n)$.

Moreover $\inf f_n \leq f_m$ for any m, which implies that

$\bar{I}(\inf_{n>m} f_n) \leq \bar{I}(f_m)$ for any m.

Hence $\bar{I}(\inf_{n>m} f_n) \leq \inf_{n>m} \bar{I}(f_n)$, that is, this corollary holds.

On the other hand, using the **theoream 4.1**, we have the following property.

Corollary 4.3. Let (f_n) be a monotone increasing sequence in L^* which converges to a limit f in R^X. Then f is in L^* if and only if $\lim \bar{I}(f_n) < \infty$.

Proof) In any case $f_n \leq f$ for every n, if $f \in L^*$ then $\bar{I}(f_n) \leq \bar{I}(f)$, hence $\lim \bar{I}(f_n) < \infty$.

On the other hand, suppose that $\lim \bar{I}(f_n) < \infty$. By the **theorem 4.1**, $\bar{I}(f) = \lim \bar{I}(f_n) < \infty$, that is, $f \in L^*$.

References

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論 文 抄 錄

本 論文은 다니엘 積分에서의 上積分에 관한 다음의 性質을 証明하였다.

1. 上積分 可能한 函数空間 $L^* = [f | \bar{I}(|f|) < \infty]$ 가 完備이다.
2. I (l, l) 가 L 에서 R로의 연속함수이다.
3. 単調収斂 定理와 Fatou 定理가 $\bar{I}$에 対하여도 成立한다.