Quotients of Matrix Semiring

Sung-ho Yang

행열 반환의 몫

梁 成 豪

Summary

This paper considers the quotient structure of matrix semirings. We prove if R is a semiring and I is a Q-ideal of R, then the set of nxn matrices over R, $M_n[R]$, is a semiring, $M_n[I]$ is a $M_n[Q]$-ideal of $M_n[R]$ and $M_n[R]/M_n[I]$ is isomorphic to $M_n[R/I]$.

Preliminaries

When R is a semiring and I is an ideal of R, the collection $\{x+I\}x\in R$ of sets $x+I=\{x+i|i\in I\}$ need not be a partition of R. P. J. Allen [1] defined Q-ideal and maximal homomorphism and established the Fundamental Theorem of Homomophisms in a large class of semirings.

The purpose of this paper is to build the quotient structure in matrix semirings.

The definitions of semiring, Q-ideal and maximal homomorphism used in [1] will be used throughout this paper. These definitions are given as follows.

Definition 1. A non-empty set R together with two associative binary operations called addition and multiplication(denoted by + and · respectively) will be called ***a semiring*** provided ;

(1) addition is a commutative operation,

(2) there exist $0\in R$ such that $x+0=x$ and $x0=0x=0$ for each $x\in R$ and

(3) multiplication distributes over addition both from the left and from the right.

Definition 2. A non-empty subset I of a semiring R will be called ***an ideal*** if a, $b\in I$ and $r\in R$ implies $a+b\in I$, $ra\in I$ and $ar\in I$.

Definition 3. A mapping φ from the semiring R into the semiring R' will be called ***a homomorphism*** if $\varphi(a+b)=\varphi(a)+\varphi(b)$ and $\varphi(ab)=\varphi(a)\varphi(b)$ for each a, $b\in R$. An ***isomorphism*** is one-to-one homomorphism. The semirings R and R' will be called ***isomorphic***(denoted by $R\cong R'$) if there exists an isomorphism from R onto R'.

Definition 4. An ideal I in the semiring R will be called ***a Q-ideal*** if there exists a subset Q of R satisfying the following conditions ;

(1) $\{q+I\}_{q\in Q}$ is a partition of R and

(2) if q_1, $q_2\in Q$ such that $q_1\neq q_2$, then $(q_1+I)\cap(q_2+I)=\varnothing$.

Definition 5. A homomorphism φ from the semiring R onto the semiring R' is said to be ***maximal*** if for each $a\in R'$ there exists $c_a\in\varphi^{-1}(\{a\})$ such that $x+\ker\varphi\subset c_a+\ker\varphi$ for each $x\in\varphi^{-1}(\{a\})$, where $\ker\varphi=\{x\in R|\varphi(x)=0\}$.

Lemma 6. Let I be a Q-ideal in the semiring R.

If $x\in R$, then there exists a unique $q\in Q$ such that $x+I\subset q+I$.

Theorem 7. If I is a Q-ideal in the semiring R, then $R/I=(\{q+I\}_{q\in Q}, \oplus_Q, \odot_Q)$ is a semiring.

Theorem 8. If φ is a maximal homomorphism from the semiring R onto the semiring R', then $R/\ker\varphi\cong R'$.

The quotient of matrix smirings

Throughout this section, unless otherwise states, R will be a commutative semiring and $M_n[R]$ will be the semiring of nxn matrices over R.

Theorem 9. If R is a semiring, then $M_n[R]$ is also.

Proof. We define the binary operations in $M_n[R]$ as follows,

$[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$ and $[a_{ij}][b_{ij}]=[\sum_{k=1}^{n}a_{ik}b_{kj}]$

for all $[a_{ij}]$, $[b_{ij}]\in M_n[R]$.

Then it is easy to show that $M_n[R]$ is a semiring.

Corollary 10. If R is a semiring and I is a Q-ideal of R, then $M_n[R/I]$ is a semiring.

Proof. By Theorem 7 and Theorem 9, it is obvious.

In this corollary, the binary operations are defined as follows ;

(1) $[q'_{ij}+I]+[q''_{ij}+I]=[q_{ij}+I]$

where $q'_{ij}+q''_{ij}+I\subset q_{ij}+I$ for all $(i, j)\in \bar{n}\times\bar{n}$

(2) $[q'_{ij}+I][q''_{ij}+I]=[q_{ij}+I]$

where $\sum_{k=1}^{n}q'_{ik}q''_{kj}+I\subset q_{ij}+I$ for all $(i,j)\in\bar{n}\times\bar{n}$.

Theorem 11. If R is a semiring and I is a Q-ideal in R, then $M_n[I]$ is a $M_n[Q]$-ideal in $M_n[R]$.

Proof. It is clear that $M_n[I]$ is an ideal in $M_n[R]$.

(1) In this theorem, $M_n[Q]$ denotes the set of all matrices over the set Q.

Suppose $[a_{ij}]\in M_n[R]$. Since $a_{ij}\in R$ for all $(i, j)\in\bar{n}\times\bar{n}$, where $\bar{n}=\{1, 2,\cdots,n\}$ and I is a Q-ideal in R, $a_{ij}\in\bigcup_{q\in Q}\{q+I\}$ for all $(i, j)\in\bar{n}\times\bar{n}$.

i.e. $a_{ij}=q_{ij}+m_{ij}$ for some $q_{ij}\in Q$ and some $m_{ij}\in I$ and for all $(i, j)\in\bar{n}\times\bar{n}$.

Thus $[a_{ij}]=[q_{ij}+m_{ij}]=[q_{ij}]+[m_{ij}]\in P+M_n[I]$ for some $P=[q_{ij}]\in M_n[Q]$.

Hence $[a_{ij}]\in\bigcup_{P\in M_n[Q]}(P+M_n[I])$.

(2) Let $[p_{ij}]$ and $[s_{ij}]$ be in $M_n[Q]$ and let $[p_{ij}]\neq[s_{ij}]$.

Then p_{ij}, $s_{ij}\in Q$ for all $(i, j)\in\bar{n}\times\bar{n}$ and $p_{ij}\neq s_{ij}$ for some $(i, j)\in\bar{n}\times\bar{n}$.

Since I is a Q-ideal in R, $(p_{ij}+I)\cap(s_{ij}+I)=\varnothing$.

i.e. $p_{ij}+m\neq s_{ij}+m'$ for all m, $m'\in I$.

Consequently, the ij-entry of every matrix in $[p_{ij}]+M_n[I]$ is different from the ij-entry of every matrix in $[s_{ij}]+M_n[I]$.

i.e. $([p_{ij}]+M_n[I])\cap([s_{ij}]+M_n[I])=\varnothing$.

Hence $M_n[I]$ is a $M_n[Q]$-ideal in $M_n[R]$.

Corollary 12. If R is a semiring and I is a Q-ideal in R, then $M_n[R]/M_n[I]=(\{P+M_n[I]\}_{P\in M_n[Q]}, \oplus_{M_n[Q]}, \odot_{M_n[Q]})$ is a semiring.

Proof. This corollary is the immediate result of Theorem 11 and Theorem 7.

The operations $\oplus_{M_n[Q]}$ and $\odot_{M_n[Q]}$ in $M_n[R]/M_n[I]$ are as follows ;

(1) $(P'+M_n[I])\oplus_{M_n[Q]}(P''+M_n[I])=P+M_n[I]$

where $P'+P''+M_n[I]\subset P+M_n[I]$ and

(2) $(P'+M_n[I])\odot_{M_n[Q]}(P''+M_n[I])=P+M_n[I]$

where $P'P''+M_n[I]\subset P+M_n[I]$.

Proposition 13. If I is a Q-ideal in a semiring R, then I is a zero-element in R/I.

Proof. Let $q^*\in Q$ such that $I\subset q^*+I$.
Then q^*+I is a zero-element in R/I by Theorem 8 in [1].
Since $0\in I\subset q^*+I$, $0=q^*+i$ for some $i\in I$.
Thus $q^*+I=q^*+0+I=q^*+q^*+i+I\subset q^*+q^*+I$.
Since q^*+q^*+I is contained in a unique coset $q'+I$ where $q'\in Q$, $q'+I=q^*+I$. *i.e.* $q^*+q^*+I=q^*+I$.
Thus $q^*+q^*=q^*+i_1$ for some $i_1\in I$. Hence $q^*+I=q^*+0+I=q^*+q^*+i+I=q^*+i_1+i+I=0+i_1+I\subset I$.
Therefore $q^*+I=I$.

Proposition 14. A Q-ideal I of semiring R is a k-ideal of R.

Proof. Recall that an ideal I is *k-ideal* if $x+i\in I$, where $x\in R$ and $i\in I$, implies $x\in I$.
Suppose $x+i\in I$, where $x\in R$ and $i\in I$. Then there exists a unique coset $q+I$ such that $x+I\subset q+I$.
Thus $x+i\in q+I$. Since $x+i\in I=q^*+I$, $x+i\in q^*+I$.
Hence $q=q^*$. Therefore $x\in x+I\subset q+I=I$.

Theorem 15. If R is a semiring and I is a Q-ideal in R, then $M_n[R]/M_n[I]$ is isomorphic to $M_n[R/I]$.

Proof. For each $a_{ij}\in R$, there exists a unique $q_{ij}\in Q$ such that $a_{ij}+I\subset q_{ij}+I$ by Lemma 6.
Define the map $\varphi: M_n[R]\to M_n[R/I]$ by $\varphi([a_{ij}])=[q_{ij}+I]$ for each $[a_{ij}]\in M_n[R]$, where $a_{ij}+I\subset q_{ij}+I$ for each $(i, j)\in \bar{n}\times\bar{n}$. Then it is clear that φ is a homomorphism from the semiring $M_n[R]$ onto $M_n[R/I]$. $\ker\varphi=M_n[I]$ by Proposition 14.
For each $[q_{ij}+I]\in M_n[R/I]$, $[q_{ij}]\in\varphi^{-1}([q_{ij}+I])$.
If $[r_{ij}]\in\varphi^{-1}([q_{ij}+I])$, then $r_{ij}+I\subset q_{ij}+I$ for all $(i, j)\in\bar{n}\times\bar{n}$.
Thus $[r_{ij}]+\ker\varphi\subset[q_{ij}]+\ker\varphi$. Hence φ is a maximal homomorphism from the semiring $M_n[R]$ onto the semiring $M_n[R/I]$.
Therefore $M_n[R]/M_n[I]\cong M_n[R/I]$ by Theorem 8.

Literatures Cited

[1]. P.J. Allen, *A fundamental theorem of homomorphism for semirings*, Proc. Amer. Math. Soc. 21.(1969). 412—416.

[2]. Y.B. Chun, *Quotients of polynomial semiring*, J. of N.S.R.I. Vol. 2.(1978). Yonsei University.

〈국문초록〉

행열 반환의 몫

이 논문에서는 R이 semiring이고 I가 R에서의 Q-ideal이면 $M_n[I]$는 $M_n[R]$에서 $M_n[Q]$-ideal이 되어 $M_n[R]/M_n[I]$는 semiring이 됨을 보였고 또 $M_n[R]/M_n[I]$와 $M_n[R/I]$는 서로 동형임을 보였다.