On Ill-posed problems and Regularization Methods

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Ill—posed 문제와 調整方法에 관한 소고

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1. Introduction

The operator equation $Tx=y$ where T is a mapping some space into another has a solution if and only if y is in the range of T. This embodies the notion of a solution in the traditional sense; it is an ideal situation. On the other hand, one may look at the problem from a different angle.

In this paper we introduce the weighted generalized inverse of a linear operator in Hilbert space and we investigate the solutions of constrained minimization problem.

Let X and Y be (real or complex) Hilbert spaces and let $A: X \rightarrow Y$ be a bounded linear operator. We denote the range of A by $R(A)$, the null space of A by $N(A)$, and the adjoint of A by A^*. For any subspace S of a Hilbert space H, we denote by $S^\perp$ the orthogonal complement of S and the closure of S by $\bar{S}$. Then we have the following orthogonal decompositions of X and Y [Groetsch(1977)]:

$$X=N(A)\oplus N(A)^\perp = N(A)\oplus\overline{R(A^*)}$$

$$Y=N(A^*)\oplus N(A^*)^\perp=N(A^*)\oplus\overline{R(A)}$$

The closed range theorem holds:

$R(A)$ is closed in Y if and only if $R(A^*)$ is closed in X. Consider an operator equation of the first kind:

(1.1) $Ax=y$, $x \in X$, $y \in Y$.

Definition 1.1. For a given $y \in Y$, an element $u \in X$ is called a least squares solution of the operator equation if and only if $\|Au-y\| \leq \|Ax-y\|$ for all $x \in X$.

Definition 1.2. An element ν is called a least squares solution of minimal norm of (1.1) if and only if ν is a least squares solution of (1.1) and $\|\nu\| < \|u\|$ for all least squares solutions u of (1.1)

A least squares solution of minimal norm is also called a best approximate solution or a pseudo-solution. For each $y \in R(A)\oplus R(A)^\perp$, the set of least squares solutions is non-empty, closed, and convex. Hence there is a unique minimal norm solution.

Definition 1.3. Let A be a bounded linear operator from X into Y. The generalized inverse, denoted by A^+, is a linear operator from the subspace $R(A)\oplus R(A)^\perp$ into X, defined by $A^+ y=\nu$ where ν is the least squares solution of minimal norm of the equation $Ax=y$.

Definition 1.4. The operator equation (1.1) is said to be well-posed (relative to the spaces X and Y) if for each $y \in Y$, (1.1) has a unique best

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approximate solution which depends continuously on Y; otherwise the equation is said to be ill-posed.

Note: when A is a linear operator with inverse, then $A^+=A^{-1}$ and the least squares solution of minimal norm coincides with the exact solution.

Theorem 1.5. Let $A:X \to Y$ be a bounded linear operator. Then the following statements are equivalent:

(a) The operator equation (1.1) is well-posed.

(b) A has a closed range in Y.

(c) A^+ is a bounded linear operator on Y into X.

Proof) (b) $\Leftrightarrow$ (c): The proof is in the Groetsch [1977]. (a) $\Leftrightarrow$ (b): If A has a closed range, then $Y=R(A)\oplus R(A)^\perp=D(A^+)$, where $D(A^+)$ is the domain of A^+

Thus we know that (a), (b), (c) are equivalent.

Remarks. (1) According to theorem 1.5, if the range of A is closed, then the operator equation is well-posed and A^+ is defined on all of Y, since $R(A)=\overline{R(A)}$. If R(A) is not closed, then the operator equation (1.1) is ill-posed and A^+ is an unbounded densely defined operator.

(2) For $y \in D(A^+)$, $A^+y \in N(A)^\perp$ and the set of all least squares solutions S is a nonempty closed convex set:

$S=\{u: u=A^+y+v \text{ for } v \in N(A)\}$

(3) Thus, for $y \in D(A^+)$, the least squares solution of minimal norm $\bar{u}$ of the operator equation (1.1) is the least squares solution which lies in $N(A)^\perp$.

2. Existence and Uniqueness of the solution of the problem

Let $L:T \to Z$ be a bounded linear operator, where Z is a Hilbert space. We assume that the range R(L) of L is closed in Z, but the range R(A) of A is not necessarily closed in Y. We consider the following minimization problem: Let $S_z=\{x \in X: x \text{ is a least squares solution of } Lx=z,\ z \in Z\}$

Then the problem is to find $w \in S_z$ such that

(2.1) $\|Aw-y\| \leq \|Ax-y\|$ for all $x \in S_z$.

In this section we state the conditions under which the solution of the problem (2.1) exists and is unique. Since for any $u \in S_z$, $u=L^+z+v$ for some $v \in N(A)$, the constrained minimization problem (2.1) is equivalent to

$\inf\{\|Ax-y\|: x \in S_z\}$

$=\inf\{\|A(L^+z+x_1)-y\|: x_1 \in N(L)\}$

$=\inf\{\|u-y\|: u \in AS_z\}$.

Note that AS_z is a translate of the subspace AN(L). Thus the problem has a solution for every $y-A(L^+z) \in AN(L)$ if and only if AN(L) is closed, and the solution is unique if and only if $N(A)\cap N(L)=\{0\}$.

Throughout this paper, we assume that $N(A)\cap N(L)=\{0\}$ and AN(L) is closed, i.e. that the constrained minimization problem (1.1) has a solution for each $y-A(L^+Z) \in D(A_L^+)$ and the solution is unique.

Proposition 2.1 Suppose that $T:X \to Y$ is a bounded linear operator and let P be the projection of Y onto $\overline{R(T)}$, then the following conditions on $u \in X$ are equivalent:

(a) $Tu=Pb$.

(b) $\|Tu-b\| \leq \|Tx-b\|$ for all $x \in X$.

(c) $T^*Tu=T^*b$.

Proof) See Groetsch (1977).

We define a new inner product in X:

(2.2) $[u,v]=\langle Au,Av\rangle_Y+\langle Lu,Lv\rangle_Z$ for $u,v \in X$.

Let $M=\{x \in X: A^*Ax-A^*y \in N(L)^\perp\}$.

Then the following proposition is an immediate consequence of the definition of $[\cdot,\cdot]$ and the assumption that $N(A)\cap N(L)=\{0\}$.

Proposition 2.2 (a) The equation (2.2) defines an inner product in X.

(b) M is a closed subspace of X and is the orthogonal complement of N(L) with respect to the new inner product, i.e., $X=N(L)\oplus_1 M$.

Proof) (a) It is easy and omitted.

(b) For every $x \in \bar{M}$ there is a sequence (x_n) in M such that $\lim x_n=x$. Hence $Ax_n \to Ax$ since A is a bounded linear operator.

Thus, for all $u \in N(L)$, $[u, A^*Ax_n - A^*y] = 0$ if and only if $\lim_{n\to\infty} [Au, A^*Ax_n - A^*y] = [Au, Ax-y] = 0$ Namely, $A^*Ax - A^*y \in N(L)^\perp$

Since $x \in \bar{M}$ was arbitrary, M is closed and so $X = N(L) \oplus_L M$.

Theorem 2. 3 An element $w \in X$ is a solution to the problem (1.1) if and only if $A^*Aw - A^*y \in N(L)^\perp$ and $L^*L_w = L^*z$.

Proof) By proposition 2.1, $w \in S_z = \{x \in X : x$ is a least squares solution of $Lx = z,\ z \in Z\}$ if and only if $L^*Lw = L^*z$.

Let $w \in S_z = \{L^+z + s : s \in N(L)\}$ such that

$\| Aw - y \| \le \| Ax - y \|$ for all $x \in S_z$

Then $\| A(L^+z+s) - y \| < \| A(L^+z+x) - y \|$ for all $x \in N(L)$, where $w = L^+z + s$.

Since $Y = \overline{R(A_L)} \oplus R(A_L)^\perp$ where A_L denote the restriction of A onto N(L). $As - \{y - A(L^+z)\} \in R(A_L)^\perp$

Thus, for all $x \in N(L)$, $(Ax, As - \{y - A(L^+z)\}) = 0$ if and only if $(x, A^*As - A^*\ y - A(L^+z)) = 0$ for all $x \in N(L)$. Hence $A^*Aw - A^*Ay \in N(L)^\perp$

By this theorem, the problem of constrained minimization (2.1) is equivalent to finding an element $w \in M$ such that $L^*Lw = L^*z$. Thus the solution w is the least squares solution of X_L-minimal norm of the equation (1.1).

3. Regularization, Existence and Uniqueness of the Regularized Solution.

When the range of A is closed, the problem (2.1) is well-posed. Hence our interest is in the case that the range of A is not closed and therefore the problem is ill-posed.

Instead of solving this ill-posed problem directly we will regularize it by a family of stable minimization problems.

Let W be the product space of Y and Z with the usual inner product : $W = Y \times Z$

$\langle (y_1, z_1), (y_2, z_2) \rangle_W = \langle y_1, y_2 \rangle_Y + \langle z_1, z_2 \rangle_Z$

for $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$.

We drop the subscripts X, Y and Z for the inner product and norms whenever the meaning is clear from the context.

For $\alpha > 0$, let C_α be a linear operator from X into W defined by $C_\alpha x = (Ax, \sqrt{\alpha} Lx)$ for $x \in X$.

Lemma 3.1 For $\alpha > 0$, the range $R(C_\alpha)$ of C_α is closed if R(L) and A(N(L)) are closed.

Proof) See to Song (1978).

Corollary 3.2 Suppose that R(L) and A(N(L)) are closed. Suppose that $N(A) \cap N(L) = \{0\}$. Let $b = (y, 0)$ in W. Then, for $\alpha > 0$, the operator $C_\alpha x = b$ is well-posed.

Proof) See to Song(1978).

We denote by U_α the unigue least squares solution of minimal norm of the equation $C_\alpha x = b$ for each $\alpha > 0$. That is, $U_\alpha = C_\alpha^{\ +} x = b$.

From the definition of C_α and inner product of W,

$C_\alpha x - b = (Ax, \sqrt{\alpha}\ Lx) - (y, 0) = (Ax - y, \sqrt{\alpha}\, Lx)$

and $\| C_\alpha x - b \|^2$

$= \langle C_\alpha x - b, C_\alpha x - b \rangle$

$= \langle Ax - y, Ax - y \rangle + \alpha \langle Lx, Lx \rangle$

$= \| Ax - y \|^2 + \alpha \| Lx \|^2$

Let us write $J_\alpha(x) = \| Ax - y \|^2 + \alpha \| Lx \|^2$

Theorem 3.3 Let $\alpha > 0$. An element x_α in X minimizes the quadratic functional $J_\alpha(x)$ if and only if $(A^*A + \alpha L^*L)x_\alpha = A^*y$.

Proof) An element x_α in X minimizes the quadratic functional $J_\alpha(x)$ if and only if

$J_\alpha(x) = 2(A^*Ax - A^*y) + 2(L^*Lx_\alpha) = 0$, i.e., $(A^*A + \alpha L^*L)x_\alpha = A^*y$.

We can approximate least squares solutions by applying the steepest descent method.

The method of steepest descent for minimizing $J_\alpha(x)$ is given by $x_{n+1} = x_n - \alpha_n r_n$, where $r_n = C_\alpha^* C_\alpha x_n - C^* b$ and

$$\alpha_n = \frac{\| r_n \|^2}{\| C_\alpha r_n \|^2}.$$

The sequence generated by steepest descent method converges to an element $u \in S_\alpha = \{z : \inf \| C_\alpha x - b \| = \| C_\alpha z - b \| \}$. $\{x_n\}$ converges to u_α if and only if $x_0 \in R(C_\alpha^*)$ for any initial approximation $x_0 \in X$.

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국 문 초 록

본 논문에서는 Hilbert 공간상에서 제한된 선형연산자의 minimization 문제를 조사하는 과정에서 그 연산자가 ill-posed인 경우 해의 존재성을 논하였다.