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ORE INVARIANT OF ARTINIAN RINGS

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For a ring A with a derivation δ, the Ore extension of derivation type $A[x;\delta]$ is the free A-module with basis $\{1, x, x^2, \dots\}$ and with the multiplication extended from A by $xa = ax + \delta(a)$. We say that the ring A is Ore invariant if whenever $A[x;\delta] \cong B[y;\varepsilon]$ we have $A \cong B$.

Armendariz, Koo and Park [1] have considered Ore extension isomorphism problem. Actually they have observed that every (von Neumann) regular self-injective ring with no $\mathbf{Z}$-torsion is Ore invariant. But still it has been remained open for the Ore invariant property of Artinian rings.

We investigate in this paper the Ore invariant property of a certain class of Artinian rings. First, we consider the corresponding relationship of prime ideals between rings A and B whenever A is an Artinian ring and $A[x;\delta] \cong B[y;\varepsilon]$. With using this result, we prove the Ore invariant property of a certain class of Artinian rings. Explicitly, we show that every Artinian ring with no $\mathbf{Z}$-torsion, for which the index of nilpotency of prime factor rings is constant, has always the Ore invariant property.

Now we introduce standard fact about prime radical $P(A)$ and Jacobson radical $J(A)$ of a ring A.

PROPOSITION 1. *If A is a right Artinian ring, then $J(A) = P(A)$.*

From the Hopkins' Theorem together with [3, Theorem 15.19], we have following remarkable fact.

PROPOSITION 2. *Every right Artinian ring is right Noetherian.*

In the ring $\mathbf{Z}$ of integers, there are infinitely many prime ideals which is called as Euclid's Theorem. But for Artinian ring case, the following result can be easily checked.

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PROPOSITION 3. *In a right Artinian ring, there are only finitely many prime ideals.*

DEFINITION 4. For a ring A, a map δ of A is called a *derivation of A* if for any a and b in A,

$$\delta(a+b) = \delta(a) + \delta(b) \quad \text{and}$$
$$\delta(ab) = a\delta(b) + \delta(a)b.$$

For a standard example of a derivation, let a be an element of A. Then $\delta_a(x) = xa - ax$ for x in A, is surely a derivation of A. This particular derivation is called *the inner derivation of A induced by a*.

For a ring A with a derivation δ, we can construct a ring $A[x;\delta]$ which is called an *Ore extension of A of derivation type*. In fact $A[x;\delta]$ is the ring of polynomials with coefficients in A. Addition is the usual addition of polynomials and multiplication is formed by the rule

$$xa = ax + \delta(a) \quad \text{for } a \text{ in } A.$$

For a ring A, we say that A is *of no $\mathbb{Z}$-torsion* if $na = 0$ for $n \neq 0$ in $\mathbb{Z}$ implies $a = 0$.

DEFINITION 5. Let A be a ring with a derivation δ. A *δ-ideal* (or *δ-invariant ideal*) of A is any two-sided ideal I of A such that $\delta(I) \subseteq I$. If A is nonzero and the only δ-ideals in A are 0 and A, then A is said to be *δ-simple*. A *δ-prime ideal* of A is any proper δ-ideal P such that whenever I and J are δ-ideals of A with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. If 0 is a δ-prime ideal of A, then A is said to be a *δ-prime ring*. Given any two-sided ideal K of A, we define $(K;\delta) = \{r \in A | \delta^n(r) \in K \text{ for all } n = 0, 1, 2, \ldots\}$, which is the largest δ-ideal of A contained in K. Note that if K is a prime ideal , then $(K;\delta)$ is a δ-prime ideal.

PROPOSITION 6. *Let A be a right Artinian ring with a derivation δ. If A has no $\mathbb{Z}$-torsion, then every prime ideal is a δ-ideal.*

Proof. In such a right Artinian A, let P be a prime ideal of A. Then by [5, Proposition 1.1], $(P;\delta)$ is a prime ideal. But since $(P;\delta) \subseteq P$ and $(P;\delta)$ is maximal, we have $(P;\delta) = P$. This means that P is a δ-ideal.

As in the ordinary polynomial ring case, it can be easily checked that Hilbert Basis Theorem is also valid for the Ore extension case:

PROPOSITION 7. *Let A be a right Noetherian ring with a derivation δ. Then the Ore extension $A[x;\delta]$ is also right Noetherian.*

The conditions which we shall now impose on our rings in order to insure the existence of idempotent elements in every idempotent coset modulo the Jacobson radical are given in the following due to I. Kaplansky.

DEFINITION 8. A ring A having the Jacobson radical $J(A)$ is called an *SBI ring* (suitable for building idempotent elements) if and only if

(1) the equation $x^2 - x = z$ where $z \in J(A)$ has a solution $z_1 \in J(A)$ such that
(2) the subring of A of elements commuting with z coincides with the subring of elements commuting with z_1.

PROPOSITION 9. *Every right Artinian ring is an SBI ring.*

Proof. By [3, Theorem 15.19], the Jacobson radical of a right Artinian ring is nilpotent. So the Jacobson radical is nil. Therefore by [7, Proposition III.8.3], every right Artinian ring is SBI.

MAIN THEOREMS

We introduce a notation: For a ring A, $Spec(A)$ denotes the set of all prime ideals of A. As we have shown, if A is right Artinian, then A has only finitely many prime ideals and so $Spec(A)$ is a finite set.

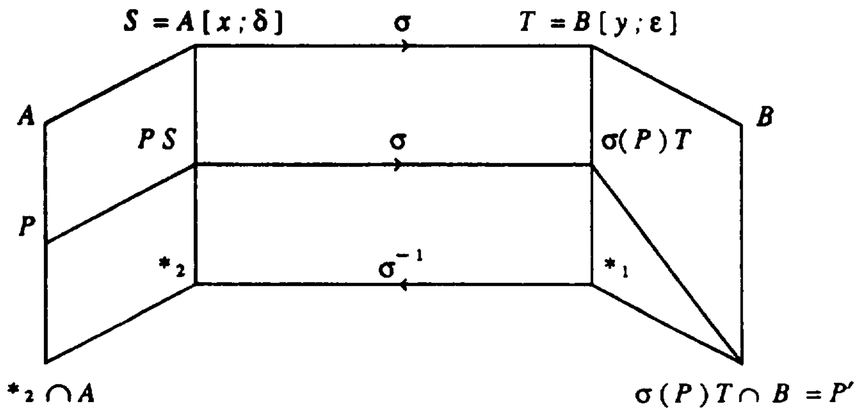
THEOREM 10. *Suppose A is a right Artinian ring with no $\mathbb{Z}$-torsion. If $A[x;\delta] \overset{\sigma}{\cong} B[y;\varepsilon]$, then we have the following :*

(1) *There is an one-to-one correspondence between $Spec(A)$ and $Spec(B)$, via $P \mapsto \sigma(P)B[y;\varepsilon] \cap B$ and $Q \mapsto \sigma^{-1}(Q)A[x;\delta] \cap A$ for $P \in Spec(A)$ and $Q \in Spec(B)$. In addition,*

$$A/P \cong B/\sigma(P)B[y;\varepsilon] \cap B.$$

(2) *B is also a right Artinian ring with no $\mathbb{Z}$-torsion.*

Proof. (1) For the convenience of our notation, let $S = A[x;\delta]$ and $T = B[y;\varepsilon]$. For our visualization, consider following diagram:



Let P be a prime ideal of A. Then by Proposition 6, P is a δ-ideal, i.e., P is a δ-invariant ideal of A. So PS is a two-sided ideal of $S = A[x;\delta]$ by [8, Lemma 1.3]. Now since σ is a ring isomorphism, $\sigma(PS) = \sigma(P)T$ is also a two-sided ideal of T. Moreover since P is prime and δ-invariant, PS is also a prime ideal of S by [8, Lemma 1.3] and hence $\sigma(P)T$ is a prime ideal of T. Thus $\sigma(P)T \cap B = P'$ is a ε-prime ideal of B by [8, Lemma 1.3]. On the other hand by the fact that A is $\mathbb{Q}$-algebra, B is also $\mathbb{Q}$-algebra and so for any prime ideal Q of B, B/Q is also a $\mathbb{Q}$-algebra. Thus by [5, Corollary 1.4], every ε-prime ideal of B is a prime ideal. By this fact, ε-prime ideal $\sigma(P)T \cap B = P'$ of B is a prime ideal. Let $*_1 = P'T$. Then $*_1$ is a prime ideal of T by [8, Lemma 1.3] and so $*_2 = \sigma^{-1}(*_1)$ is also a prime ideal of S. Thus by [8, Lemma 1.3], $*_2 \cap A$ is a δ-prime ideal of A. But as we already noted, since every δ-prime ideal of A is a prime ideal, we have that $*_2 \cap A$ is a prime ideal of A. Now since A is right Artinian, $*_2 \cap A$ is a maximal ideal by Wedderburn-Artin Theorem. So from the fact $*_2 \cap A \subseteq P$, we have that $*_2 \cap A = P$ and so $PS = *_2$. Moreover we have $\sigma(P)T = *_1 = P'T$. Since σ is a ring isomorphism from S to T, we may have the naturally induced isomorphism $S/PS \cong T/\sigma(P)T = T/P'T$. Now since P is δ-invariant, we may naturally induce a derivation $\bar{\delta}$ on A/P as in [8, Lemma 1.4]. In fact, $\bar{\delta}(a+P) = \delta(a) + P$ for any $a+P$ in A/P. Similarly we also can induce a derivation $\bar{\varepsilon}$ on B/P' because P' is ε-invariant. So from [8, Lemma 1.4] or from the fact $S/PS \cong T/P'T$, we have the naturally induced isomorphism $(A/P)[x;\bar{\delta}] \cong (B/P')[y;\bar{\varepsilon}]$ from the isomorphism σ. But since the ring A/P is simple Artinian with no $\mathbb{Z}$-torsion, $A/P \cong B/P'$

by [2, Lemma 10]. Moreover, $P' = \sigma(P)T \cap B$ is a maximal ideal of B because B/P' is simple. Now let Q be a prime ideal of B. Since B has no $\mathbb{Z}$-torsion, $(Q;\varepsilon)$ is prime and an ε-ideal (i.e. ε-prime ideal) of B by [5, Proposition 1.1]. Let $Q_0 = (Q;\varepsilon)$. Then Q_0T is a prime ideal of T by [8, Lemma 1.3]. So, $\sigma^{-1}(Q_0T) = \sigma^{-1}(Q_0)S$ is also a prime ideal of S. Thus $Q' = \sigma^{-1}(Q_0)S \cap A$ is δ-prime by [8, Lemma 1.3]. So, Q' is also a prime ideal of A since A is a right Artinian ring with no $\mathbb{Z}$-torsion. Since $\sigma(Q')T \cap B$ is maximal in B by the preceding part of our proof, $\sigma(Q')T \cap B = Q_0 = Q$ from the fact that $\sigma(Q')T \cap B \subseteq Q_0 \subseteq Q$. Hence there is an one-to-one correspondence between $Spec(A)$ and $Spec(B)$.

(2) For more information about B, note that A has only finitely many prime ideals, i.e., $Spec(A)$ is a finite set by Proposition 3. By one-to-one correspondence between $Spec(A)$ and $Spec(B)$ in (1), we have that $Spec(B)$ is also a finite set, that is, B also has a finitely many prime ideals. For our argument (2), we will claim that B is right Artinian. For the convenience let $Spec(A) = \{P_1, P_2, \ldots, P_n\}$ and $Spec(B) = \{P_1', P_2', \ldots, P_n'\}$. By the one-to-one correspondence as in (1) we may assume that each P_i is correspondent to P_i' and $A/P_i \cong B/P_i'$ as we have shown before. Now note that $J(A)$ is the prime radical, i.e., $J(A)$ is the intersection of all prime ideals of A by Proposition 1. So we have that $J(A) = P_1 \cap P_2 \cap \cdots \cap P_n$. By the one-to-one correspondence between $Spec(A)$ and $Spec(B)$ we also can check that every prime ideal of B is also a maximal ideal. Therefore each P_i' is also a maximal ideal of B. So we have $J(B) = P_1' \cap P_2' \cap \cdots \cap P_n'$. Now $A/J(A) = A/P_1 \times \cdots \times A/P_n$ by Chinese Remainder Theorem. Thus

$$\begin{aligned} A/J(A) &= A/P_1 \times \cdots \times A/P_n \\ &\cong B/P_1' \times \cdots \times B/P_n' \\ &\cong B/J(B). \end{aligned}$$

On the other hand, since A is right Artinian, A is right Noetherian by Proposition 2. and so $A[x;\delta]$ is right Noetherian by Proposition 7. So $B[y;\varepsilon]$ is right Noetherian and hence B is also right Noetherian. In fact, let $I_1 \subseteq I_2 \subseteq \cdots$ be a chain of right ideals of B. Then we have an ascending chain of right ideals $I_1(B[y;\varepsilon]) \subseteq I_2(B[y;\varepsilon]) \subseteq \cdots$ of $B[y;\varepsilon]$. But since $B[y;\varepsilon]$ is right Noetherian, there is a positive integer N such that $I_N(B[y;\varepsilon]) = I_{N+1}(B[y;\varepsilon]) = \cdots$. Thus we have $I_N = I_{N+1} =$

$\cdots$. By this fact B is right Noetherian. Now since A is right Artinian, the Jacobson radical $J(A)$ is nilpotent by [3, Theorem 15.19]. Say $J(A)^N = 0$ for some positive integer N. Then $(J(A)[x;\delta])^N = J(A)^N[x;\delta] = 0$ because $J(A)$ is δ-invariant. So $J(A)[x;\delta]$ is also nilpotent. Note that

$$\begin{aligned} J(A)[x;\delta] &= (P_1 \cap \cdots \cap P_n)[x;\delta] \\ &= P_1[x;\delta] \cap \cdots \cap P_n[x;\delta] \\ &= P_1S \cap P_2S \cap \cdots \cap P_nS. \end{aligned}$$

Hence $\sigma(J(A)[x;\delta]) = \sigma(P_1S \cap \cdots \cap P_nS) = \sigma(P_1S) \cap \cdots \cap \sigma(P_nS) = P_1'T \cap \cdots \cap P_n'T$. By observing that $J(B) = P_1' \cap \cdots \cap P_n' \subseteq P_1'T \cap \cdots \cap P_n'T$, we can see that $J(B)$ is also nilpotent because $\sigma(J(A)[x;\delta]) = P_1'T \cap \cdots \cap P_n'T$ is nilpotent. So B is right Noetherian, $B/J(B) = A/J(A)$ is semisimple Artinian and $J(B)$ is nilpotent. Therefore by C. Hopkins' Theorem, B is right Artinian. Obviously, B has no $\mathbb{Z}$-torsion.

COROLLARY 11. *Let A be a semisimple Artinian ring with no $\mathbb{Z}$-torsion. If $A[x;\delta] \cong B[y;\varepsilon]$, then $A \cong B$.*

Proof. Let $Spec(A) = \{P_1, \dots, P_n\}$ and $Spec(B) = \{P_1', \dots, P_n'\}$ and we may assume that each P_i is correspondent to P_i' and $A/P_i \cong B/P_i'$ by Theorem 10.-(1). Since A is semisimple, $J(A) = P_1 \cap \cdots \cap P_n = 0$. Also we have $J(B) = P_1' \cap \cdots \cap P_n' = 0$. We can get $A/J(A) \cong B/J(B)$ as in the proof of Theorem 10.-(2). Thus we have

$$A = A/J(A) \quad \cong \quad B/J(B) = B.$$

THEOREM 12. *Assume that A is a right Artinian ring with no $\mathbb{Z}$-torsion of which every prime factor ring of A has constant index of nilpotency. If $A[x;\delta] \cong B[y;\varepsilon]$, then $A \cong B$.*

Proof. Since A is right Artinian, A has only finitely many prime ideals by Proposition 3. Since in a right Artinian ring prime ideal and maximal ideal are exactly coincided, A has also only finitely many maximal ideals, say $P_1, \dots, P_k$. So by assumption and [4, Theorem 7.12], there is a positive integer n such that for every $i = 1, \dots, k$, we have

$$A/P_i = \mathrm{Mat}_n(D_i)$$

for some division ring D_i. So we have

$$\begin{aligned} A/J(A) &= A/P_1 \times \cdots \times A/P_k \\ &= \mathrm{Mat}_n(D_1) \times \cdots \times \mathrm{Mat}_n(D_k) \\ &= \mathrm{Mat}_n(D_1 \times \cdots \times D_k). \end{aligned}$$

Thus by Proposition 9. and [7, Theorem III.8.1], $A = \mathrm{Mat}_n(R)$ for some ring R with $R/J(R) = D_1 \times \cdots \times D_k$. Denote $\bar{R} = R/J(R)$. Then $A/J(A) = \mathrm{Mat}_n(\bar{R})$. Since A is right Artinian, $J(A)$ is nilpotent by [3, Theorem 15.19]. Also $J(A)$ is δ-invariant by Proposition 6. and hence $J(A)[x;\delta]$ is nilpotent. Now note that every nilpotent ideal is contained in the Jacobson radical [3, Corollary 15.10]. Thus $J(A)[x;\delta] \subseteq J(A[x;\delta])$.

Now let $\{e_{ij}\}_{i,j=1}^n$ and $\{f_{ij}\}_{i,j=1}^n$ be two sets of matrix units of the ring $A[x;\delta] = \mathrm{Mat}_n(R)[x;\delta]$. Then by [2, Theorem 2], we may identify $\mathrm{Mat}_n(R)[x;\delta] \equiv \mathrm{Mat}_n(R[x;\delta])$. Now since the Jacobson radical $J(R)$ of R is δ-invariant, we may naturally induce a derivation $\bar{\delta}$ on $\bar{R} = R/J(R)$ via $\bar{\delta}(a + J(R)) = \delta(a) + J(R)$. So from the ring $R[x;\delta]$ we can consider naturally the induced ring structure on $\bar{R}\left[x;\bar{\delta}\right] = (R/J(R))\left[x;\bar{\delta}\right]$ which is actually isomorphic to $R[x;\delta]/J(R)[x;\delta]$ by [8, Lemma 1.4]. Now let $\bar{\ } : \mathrm{Mat}_n(R[x;\delta]) \longrightarrow \mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$ be the canonical ring epimorphism. Then of course $\{\bar{e}_{ij}\}_{i,j=1}^n$ and $\{\bar{f}_{ij}\}_{i,j=1}^n$ are also two sets of matrix units of the ring $\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$. Observe that $\bar{R}$ is an Abelian regular ring by [4, Theorem 3.2]. So there exists a matrix M in $\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$ such that

$$\bar{f}_{ij} = M^{-1}\bar{e}_{ij}M \quad \text{for} \quad i,j = 1,\ldots,n$$

by [1, Corollary 8]. In particular, $\bar{f}_{11} = M^{-1}\bar{e}_{11}M$. Now we claim that $\bar{e}_{11}\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$ is isomorphic to $\bar{f}_{11}\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$ as modules over the ring $\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$. Note that $\bar{e}_{11}(M\bar{f}_{11})\bar{f}_{11} = M\bar{f}_{11}$, $\bar{f}_{11}(M^{-1}\bar{e}_{11})\bar{e}_{11} = M^{-1}\bar{e}_{11}$. Also $(M\bar{f}_{11})(M^{-1}\bar{e}_{11}) = \bar{e}_{11}\bar{e}_{11} = \bar{e}_{11}$ and $(M^{-1}\bar{e}_{11})(M\bar{f}_{11}) = \bar{f}_{11}\bar{f}_{11} = \bar{f}_{11}$. So by [7, Proposition III.7.4],

$$\bar{e}_{11}\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right]) \cong \bar{f}_{11}\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$$

as modules over $\mathrm{Mat}_n(\bar{R}\left[x;\bar{\delta}\right])$. Now note that $J(R)[x;\delta] \subseteq J(R[x;\delta])$ since $J(R)[x;\delta]$ is nilpotent. So there is a naturally induced ring epimorphism from $\bar{R}\left[x;\bar{\delta}\right]$ to $R[x;\delta]/J(R[x;\delta])$. By this epimorphism, we have also the induced ring epimorphism θ from $\mathrm{Mat}_n(\bar{R}[x;\bar{\delta}])$ to $\mathrm{Mat}_n(R[x;\delta]/J(R[x;\delta]))$. By [6, Theorem 1.2.6], we also have that $\mathrm{Mat}_n(R[x;\delta]/J(R[x;\delta]))$ is isomorphic to the ring $A[x;\delta]/J(A[x;\delta])$.

Now since $\bar{e}_{11}\mathrm{Mat}_n(\bar{R}[x;\bar{\delta}])$ is isomorphic to $\bar{f}_{11}\mathrm{Mat}_n(\bar{R}[x;\bar{\delta}])$ as modules over the ring $\mathrm{Mat}_n(\bar{R}[x;\bar{\delta}])$, we have by our claim

$$\theta(\bar{e}_{11})\mathrm{Mat}_n\left(\frac{R[x;\delta]}{J(R[x;\delta])}\right) \cong \theta(\bar{f}_{11})\mathrm{Mat}_n\left(\frac{R[x;\delta]}{J(R[x;\delta])}\right)$$

as modules over the ring $\mathrm{Mat}_n(R[x;\delta]/J(R[x;\delta]))$. Thus we have

$$e_{11}\mathrm{Mat}_n(R[x;\delta]) \cong f_{11}\mathrm{Mat}_n(R[x;\delta])$$

as right modules over the ring $\mathrm{Mat}_n(R[x;\delta])$ by [7, Proposition III.8.1].

Now we need a necessary argument : Assume that η is a ring isomorphism from the ring $\mathrm{Mat}_n(R[x;\delta])$ to $\mathrm{Mat}_n(C)$, where C is a ring. Let $\{u_{ij}\}_{i,j=1}^n$ and $\{v_{ij}\}_{i,j=1}^n$ be two sets of the standard matrix units of $\mathrm{Mat}_n(R[x;\delta])$ and $\mathrm{Mat}_n(C)$, respectively. If we put $g_{ij} = \eta^{-1}(v_{ij})$ then $\{g_{ij}\}_{i,j=1}^n$ is also a set of matrix units of $\mathrm{Mat}_n(R[x;\delta])$. Since both $\{u_{ij}\}_{i,j=1}^n$ and $\{g_{ij}\}_{i,j=1}^n$ are two sets of matrix units of the ring $\mathrm{Mat}_n(R[x;\delta])$, we have

$$u_{11}\mathrm{Mat}_n(R[x;\delta]) \cong g_{11}\mathrm{Mat}_n(R[x;\delta])$$

as right modules over the ring $\mathrm{Mat}_n(R[x;\delta])$ by our previous argument. Hence as rings,

$$u_{11}\mathrm{Mat}_n(R[x;\delta])u_{11} \cong g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$$

because both u_{11} and g_{11} are idempotents, and $u_{11}\mathrm{Mat}_n(R[x;\delta])u_{11}$ (resp. $g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$) is the endomorphism ring of the right module $u_{11}\mathrm{Mat}_n(R[x;\delta])$ (resp. $g_{11}\mathrm{Mat}_n(R[x;\delta])$) over $\mathrm{Mat}_n(R[x;\delta])$. But since $\{u_{ij}\}_{i,j=1}^n$ is a set of the standard matrix units of the ring $\mathrm{Mat}_n(R[x;\delta])$, we have that

$$R[x;\delta] = u_{11}\mathrm{Mat}_n(R[x;\delta])u_{11}$$

and hence

$$R[x;\delta] \cong g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}.$$

Now note that the ring $g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$ is the centralizer of $\{g_{ij}\}_{i,j=1}^{n}$ in the ring $\mathrm{Mat}_n(R[x;\delta])$. Also note that C is the centralizer of the matrix units $\{v_{ij}\}_{i,j=1}^{n}$ in the ring $\mathrm{Mat}_n(C)$ by [7, Proposition III.7.6]. If we define a map f from $g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$ to C by $f(s) = \eta(s)$ for s in $g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$. Then f is of course an isomorphism and so $g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11} \cong C$. But since $R[x;\delta] \cong g_{11}\mathrm{Mat}_n(R[x;\delta])g_{11}$, we have that $R[x;\delta] \cong C$.

Still we are proceeding on proving that A is isomorphic to B. Observe that B is right Artinian with no $\mathbb{Z}$-torsion and $A/J(A) \cong B/J(B)$ by Theorem 10. Also by Theorem 10, for any prime ideal Q of B, there corresponds a prime ideal P of A such that $B/Q \cong A/P$. Thus, for any prime ideal Q of B, we have

$$B/Q = \mathrm{Mat}_n(D)$$

for some division ring D. Therefore by [7, Theorem III.8.1],

$$B = \mathrm{Mat}_n(L)$$

for some ring L as in our previous steps. Hence we have

$$\mathrm{Mat}_n(R[x;\delta]) \cong \mathrm{Mat}_n(L[y;\varepsilon])$$

from the fact that $A[x;\delta] \cong B[y;\varepsilon]$. So by our previous argument, it follows that there is a ring isomorphism λ from $R[x;\delta]$ to $L[y;\varepsilon]$. Therefore this ring isomorphism λ induces a ring isomorphism $\bar{\lambda}$ from $(R/J(R))[x;\bar{\delta}]$ to $(L/J(L))[y;\bar{\varepsilon}]$, where $\bar{\delta}$(resp. $\bar{\varepsilon}$) is the induced derivation of δ(resp. ε) to the ring $R/J(R)$ (resp. $L/J(L)$). But since $R/J(R)$ is Abelian regular, $\bar{\lambda}(R/J(R)) \cong L/J(L)$ by [1, Theorem 3]. Hence we have $\lambda(R) \subseteq L$. Similarly $\lambda^{-1}(L) \subseteq R$ and so $\lambda(R) = L$, i.e., $R \cong L$. Therefore $\mathrm{Mat}_n(R) \cong \mathrm{Mat}_n(L)$. That is $A \cong B$.

As an immediate corollary to Theorem 12, we have following interesting fact.

COROLLARY 13. *Assume that a ring A with no $\mathbb{Z}$-torsion is either local Artinian or commutative Artinian. Then if $A[x;\delta] \cong B[y;\varepsilon]$, then $A \cong B$.*

Finally we introduce an interesting example which shows that we can not get rid of the condition *with no $\mathbb{Z}$-torsion* in Theorem 12.

EXAMPLE 14. Let A be the ring $Z_2[x]/(x^2)$ with the derivation δ such that $\delta(\bar{x}) = 1$ where $\bar{x} = x + (x^2)$. Consider the Ore extension $A[y;\delta] = (Z_2[x]/(x^2))[y;\delta]$. If we set $e_{11} = \bar{x}y$, $e_{12} = \bar{x}$, $e_{21} = \bar{x}y^2 + y$, $e_{22} = 1 + \bar{x}y$, then they form a set of matrix units in $A[y;\delta]$. Now the centralizer of these matrix units in $A[y;\delta]$ is $Z_2[y^2]$. Therefore $A[y;\delta] \cong \mathrm{Mat}_2(Z_2[y^2]) \cong \mathrm{Mat}_2(Z_2)[t]$. But $\mathrm{Mat}_2(Z_2)$ is not isomorphic to $Z_2[x]/(x^2)$. Hence $\mathrm{Mat}_2(Z_2)$ is a simple Artinian ring of characteristic 2 which is not Ore invariant.

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