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On Maximal Convertible Matrices

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A square real matrix A is called convertible if there is a matrix $\hat{A}$ obtained from A by affixing $\pm$ signs to entries of A so that $\operatorname{per} A = \det \hat{A}$. A convertible (0,1)-matrix with total support is called maximal convertible if it is fully indecomposable and no matrix obtained from A by replacing $a\,0$ with $a\,1$ is convertible. In this paper, the existence of maximal convertible matrices with exactly r 1's for each integer r with $4n-4 \le r \le (n^2+3n-2)/2$ is proved.

1. INTRODUCTION

A real square matrix A is called convertible if there is a matrix $\hat{A}$ obtained from A by affixing $\pm$ signs to entries of A so that $\operatorname{per} A = \det \hat{A}$. Let E be a (0,1)-matrix of order n with $\operatorname{per} E \neq 0$ and let $M_n(E)$ denote the set of all $n \times n$ real matrices having 0's in which E has 0's (and possibly elsewhere). Suppose E is convertible and let $\hat{E}$ be a matrix obtained from E by affixing - signs to some of its 1's such that

$$\operatorname{per} E = \det \hat{E}.$$

It is not hard to see that the above equality is equivalent to the following statement

$$\operatorname{per} X = \det(\hat{E} \circ X) \qquad \text{for all} \quad X \in M_n(E)$$

where $\hat{E} \circ X$ stands for the Hadamard(entrywise) product of $\hat{E}$ and X. The $(0, 1, -1)$-matrix $\hat{E}$ is said to convert the permanent of the matrices in $M_n(E)$ into determinant. An $n \times n$ (0,1)-matrix $E = [e_{ij}]$ is said to have *total support* provided that each 1 in E enters into a nonzero term in the permanent expansion of E. Brualdi and Shader [1] proved that for a convertible (0,1)-matrix E having total support the existence of the matrix $\hat{E}$ such that $\operatorname{per} E = \det \hat{E}$ is unique upto diagonal equivalence of its zero pattern.

$A(0, 1, -1)$-matrix A is called *sign-nonsingular* [4] provided that each matrix with the same sign pattern as A is nonsingular. It is noted in [1] that the convertibility of E

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is equivalent to the sign-nonsingularity of $\hat{E}$. A graph theoretical characterization of convertible (0,1)-matrices was obtained by Little [5].

A square matrix is called *fully indecomposible* if it does not contain an $s \times t$ zero submatrix with $s + t = n$. If a square (0,1)-matrix E with total support is not fully indecomposable then there exist permutation matrices P and Q such that $PEQ = E_1 \oplus E_2$ where E_1 and E_2 are nonvacuous square matrices. Thus we see that a nonzero matrix with total support is, upto row and column permutations, a direct sum of fully indecomposable matrices which are called the *fully indecomposable components*. Hence it follows that a square (0,1)-matrix having total support is convertible if and only if all of its fully indecomposable components are. So, in most cases dealing with the convertibility of (0,1)-matrices, it suffices to consider only fully indecomposable matrices having total support.

For matrices A, B of the same size, A is said to be permutation equivalent to B if there exist permutation matrices P and Q such that $PAQ = B$. We write $A \leq B$ to denote that every entry of A is less than or equal to the corresponding entry of B. For a real matrix A, let $\pi(A)$ denote the number of positive entries of A.

Gibson [2] proved that for any $n \times n$ convertible (0,1)-matrix A,

$$\pi(A) \leq \frac{n^2 + 3n - 2}{2}$$

with equality if and only if A is permutation equivalent to the following $n \times n$ matrix

$$T_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

The matrix is maximal in the sense that no (0,1)-matrix A such that $T_n \leq A$ and $T_n \neq A$ is convertible. A convertible (0,1)-matrix E is called *extremal convertible* if no matrix obtained from E by replacing $a\,0$ with $a\,1$ is convertible [3]. A fully indecomposable extremal convertible matrix is called *maximal convertible* [1].

For a positive integer n, let J_n denote the all 1's matrix of order n and I_n the identity matrix of order n. Let μ_n denote the minimum number of 1's that a maximal convertible matrix of order n can have. Brualdi and Shader, *ibid.*, showed that $\mu_n \geq 3n$ for $n \geq 4$ and if $n \geq 5$ then the equality can hold only if each row and each column contains exactly three 1's. They also showed that the matrix

$$\begin{bmatrix} J_2 & I_2 & 0 & \cdots & 0 & 0 & 0 \\ I_2 & J_2 & I_2 & \cdots & 0 & 0 & 0 \\ 0 & I_2 & J_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & J_2 & I_2 & 0 \\ 0 & 0 & 0 & \cdots & I_2 & J_2 & I_2 \\ 0 & 0 & 0 & \cdots & 0 & I_2 & J_2 \end{bmatrix}$$

is maximal convertible. We see that this matrix has $4n-4$ 1's. In this paper we prove the existence of maximal convertible matrices with exactly r 1's for each integer r such that $4n-4 \le r \le (n^2+3n-2)/2$.

2. MAIN RESULT

We make use of the following properties of maximal convertible matrices due to Brualdi and Shader, *ibid.*, in constructing a new maximal convertible matrix of larger order from a given maximal convertible matrix.

LEMMA 1 [1, Proposition 3.1] *Let*

$$E = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$$

be a maximal convertible (0,1)*-matrix of order n. Then the matrix*

$$G = \left[\begin{array}{c|c} 1 & \mathbf{a}_1 \\ \hline 1 & \\ 0 & E \\ \vdots & \\ 0 & \end{array}\right]$$

is a maximal convertible matrix of of order $n+1$.

LEMMA 2 [1, Proposition 3.2] *Let G be a maximal convertible matrix of order n whose first column is equal to* $[1,1,0,\ldots,0]^T$. *Then the first two rows of G are identical and the matrix obtained from G by deleting row* 1 *and column* 1 *is a maximal convertible matrix.*

In [3], it is asked that whether there exist maximal convertible (0,1)-matries with exactly r 1's for each r such that $\mu_n \le r \le (n^2+3n-2)/2$. The following theorem provides a partial answer to this question.

THEOREM 1 *Let $n \ge 3$. Then for each integer r such that $4n-4 \le r \le (n^2+3n-2)/2$, there exist maximal convertible* (0,1)*-matrices of order n with exactly r 1's.*

Proof Let $n \ge 3$ be fixed. For each $k = 2,\ldots,n-1$, let $X_k = [x_{ij}]$ be the the $k \times k$ (0,1)-matrix defined by $x_{ij} = 1$ if and only if $i+j \ge k$. Let $\lambda_k = \pi(X_k)$. Then

$$\lambda_k = \frac{k^2+3k-2}{2}$$

and $\lambda_k - \lambda_{k-1} = k-1$.

Now define an $(n-1)\times(n-1)$ matrix Z_k as follows. If $k = n-1$, let $Z_k = X_k$. If $k < n-1$, then, define a sequence of matrices $Y_0, Y_1, \ldots, Y_{n-k-1}$ inductively as follows, and then let $Z_k = Y_{n-k-1}$.

Let $Y_0 = X_k$ and

$$Y_t = \begin{bmatrix} \mathbf{a}_t & 1 \\ Y_{t-1} & \mathbf{b}_t \end{bmatrix}$$

for $t = 0, 1, \ldots, n-k-1$, where $\mathbf{a}_t, \mathbf{b}_t$ are both $(k+t-1)$-vectors defined by

$$\mathbf{a}_t = \left(0, \ldots, 0, \frac{1-(-1)^t}{2}, 1\right),$$

$$\mathbf{b}_t = \left(1, \frac{1+(-1)^t}{2}, 0, \ldots, 0\right)^T.$$

Then since $X_k = Y_0$ is maximal convertible, the maximal convertibility of $Y_1, Y_2, \ldots, Y_{n-k-1} = Z_k$ follow successively from Lemma 1. We see here that $\pi(Y_0) = \lambda_k$, $\pi(Y_t) = \pi(Y_{t-1}) + 4$, $t = 0, 1, \ldots, n-k-1$, so that $\pi(Z_k) = \lambda_k + 4(n-k-1)$.

Let ℓ be an integer such that $1 \leq \ell \leq k-1$ and let

$$Z_{k,\ell} = \begin{bmatrix} 1 & \mathbf{e}_\ell^T \\ \mathbf{x}_\ell & Z_k \end{bmatrix}$$

where $\mathbf{x}_\ell$ denote the column ℓ of Z_k. Then it follows that $Z_{k,\ell}$ is an $n \times n$ maximal convertible matrix by Lemma 1 again. Let $f(k,\ell) = \pi(Z_{k,\ell})$. Then f is a function defined on the set $S = \{(k,\ell) | 2 \leq k \leq n-1, 1 \leq \ell \leq k-1\}$ and $f(k,\ell) = \pi(Z_k) + \ell + 3 = \lambda_k + 4(n-k-1) + \ell + 3 = \lambda_k + 4(n-k) + \ell - 1$.

To complete the proof, it suffices to show that $f(S)$consists of all the integers r such that $4n-4 \leq r \leq (n^2+3n-2)/2$. Notice first that $f(k,\ell)$ is increasing with respect to each of the components, and in particular $f(2,1) \leq f(k,\ell) \leq f(n-1, n-2)$ for all $(k,\ell) \in S$. Moreover, for each fixed k, we see that f increases by 1 as ℓ increases by 1, so that

$$\{f(k,\ell) | 1 \leq \ell \leq k-1\} = \{f(k,1), f(k,1)+1, \ldots, f(k,1)+k-2\}.$$

Finally observe that

$$f(2,1) = \lambda_2 + 4(n-2) + 1 - 1 = 4n-4,$$

$$f(n-1, n-2) = \lambda_{n-1} + 4\{n-(n-1)\} + (n-2) - 1 = \frac{k^2+3k-2}{2}$$

and also that $f(k-1, k-2) = f(k,1)$ for all $k = 3, \ldots, n-1$, because

$$\begin{aligned} & f(k,1) - f(k-1, k-2) \\ & \quad = \{\lambda_k + 4(n-k) + 1 - 1\} - \{\lambda_{k-1} + 4[n-(k-1)] + (k-2) - 1\} \\ & \quad = \lambda_k - \lambda_{k-1} - k - 1 = 0 \end{aligned}$$

Therefore $f(S) = \{4n-4, 4n-3, \ldots, (n^2+3n-2)/2\}$, and the proof is complete. ∎

Let $A = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$ be an $n \times n$ (0,1)-matrix. For each $k = 1, 2, \ldots, n-1$ the matrix

$$B = \begin{bmatrix} 1 & \mathbf{e}_k^T \\ \mathbf{x}_t & A \end{bmatrix}$$

obtained from A by "expanding" the k-th column is called an *expansion* of A. Expansion of a matrix can also be defined by expanding rows. For example the matrix G in Lemma 1 is an expansion of E obtained by expanding the first row of E.

In Theorem 1, we have constructed a finite sequence of maximal convertible matrices by expanding rows and columns to prove the existence of a maximal convertible matrices of order r for each $r = 4n-4, 4n-3, \ldots, (n^2+3n-2)/2$. Actually, starting with the maximal convertible matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

we can construct maximal convertible matrices of order n with exactly $4n-5$ 1's and $4n-6$ 1's. What we have seen so far tells us that $3n \le \mu_n \le 4n-6$. As it seems quite hard to find an explicit formula, in terms of n, of μ_n, we raise a problem of determining how fast μ_n grows as n does, more explicitly, a problem of determining

$$\lim_{n\to\infty} \frac{\mu_n}{n}.$$

For an $n \times n$ matrix $A = [a_{ij}]$, let $\sigma_{ij}(A)$ be defined by

$$\sigma_{ij}(A) = \sum_{k=1}^{n} a_{kj} + \sum_{\ell=1}^{n} a_{i\ell} - a_{i\ell}.$$

We call $\sigma_{ij}(A)$ the *cross-line-sum* of A at the posion (i, j). It can be easily shown that if A is a maximal convertible (0,1)-matrix of order n, then $\sigma_{ij}(A) \ge 4$ for all $i, j = 1, 2, \ldots, n$, by Lemma 2 and by the full indecomposability of A. In the following, for $\alpha, \beta \subset \{1, 2, \ldots, n\}$, let $A(\alpha \mid \beta)$ denote the matrix obtained from A by deleting the rows indexed by α and the columns indexed by β.

LEMMA 3 *Let A be a maximal convertible (0,1)-matrix of order n with positive diagonal entries. Then for every pair (i, j) with $i \ne j$, we have $\pi(A) - \pi(A(i,j \mid i,j)) \ge 8$.*

Proof Let (i, j), $i \ne j$, be given. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Let s_{ij} denote $\sigma_{ij}(A)$ for brevity. We first look at the case that either $s_{11} = 4$ or $s_{22} = 4$. We can assume, without loss of generality, that $s_{11} = 4$. Then either the first row or the first column of A has exactly two 1's. But then, by Lemma 2, $A(1|1)$ is maximal convertible so that

$$\sigma_{11}(A(1 \mid 1)) = \sum_{j=2}^{n} a_{2j} + \sum_{i=2}^{n} a_{i2} - a_{22} \ge 4.$$

From this we get

$$\pi(A) - \pi(A(1,2 \mid 1,2)) = s_{11} + \sigma_{11}(A(1 \mid 1)) \ge 8.$$

Suppose now that $s_{11} \ge 5$ and $s_{22} \ge 5$. Then

$$\pi(A) - \pi(A(1,2|1,2)) = s_{11} + s_{22} - (a_{12} + a_{21}) \ge 5 + 5 - 2 = 8,$$

and the proof is complete. ■

Now from Lemma 3 and from the fact that every cross-line-sum of a maximal convertible (0,1)-matrix is ≥ 4, the following theorem follows.

THEOREM 2 *Let A, B be maximal convertible* (0,1)*-matrices of order k and n respectively with A being a submatrix of B. If there exist maximal convertible matrices $A = A_0, A_1, \ldots, A_\ell = B$ such that A_{i-1} is a submatrix of A_i and* (*order of* A_i) − (*order of* A_{i-1}) ≤ 2 *for all* $i = 1, 2, \ldots, \ell$. *Then* $\pi(B) \geq \pi(A) + 4(n-k)$.

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