A Liquid Crystal Energy Functional on 3-Manifolds

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3차원 다양체상의 액정 범함수

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The free energy functional of a nematic liquid crystal n in $\Omega \subset R^3$ is given by

$$W(n) = \int_\Omega k_1 (\mathrm{Div}\ n)^2 + k_2 \langle n,\ \mathrm{Curl}\ n\rangle^2 + k_3 |n \times \mathrm{Curl}\ n|^2,$$

where k_i's are positive constants.

We will define a functional on a closed oriented Riemannian manifold M of dimension 3, which is an analogue of the above.

Let $T^* = T^*M$ be the cotangent bundle. The Riemannian structure and the orientation may be used to define a linear transformation, called the Hodge star poerator,

$$* : \Lambda^p T^* \rightarrow \Lambda^{n-p} T^*$$

which in terms of an orthonormal basis $\{w^1, \cdots, w^n\}$

$$*(w^{i_1} \wedge \cdots \wedge w^{i_p}) = \mathrm{sgn}(i_1, \cdots, i_p, j_{p+1}, \cdots, j_n)\, w^{j_{p+1}} \wedge \cdots \wedge w^{j_n},$$

where $i_1 < \cdots < i_p$ and $j_{p+1} < \cdots < j_n$ are complementary sets of the integers $\{1, \cdots, n\}$, and $\mathrm{sgn}(i_1, \cdots, i_p, j_{p+1}, \cdots, j_n)$ is the signature of the permutation $(i_1, \cdots, i_p, j_{p+1}, \cdots, j_n)$ of $(1, \cdots, n)$. This Hodge star is a pointwise isometry with respect to the inner products on $\Lambda^p T^*$ induced by the Riemannian structure on M, i. e.,

$$\langle \alpha, \beta \rangle_{\Lambda^p_x T^*} = \langle *\alpha, *\beta \rangle_{\Lambda^{n-p}_x T^*}$$

for any p-forms α and β at $x \in M$.

The Hodge inner product on $\Lambda^p T^*$ is the positive definite bilinear form defined by

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta$$

for any p-forms α and β. With respect to this inner product the exterior derivative $d_p : \Lambda^p T^* \rightarrow \Lambda^{p+1} T^*$ has a unique adjoint $\delta_{p+1} : \Lambda^{p+1} T^* \rightarrow \Lambda^p T^*$, given by

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$$\delta_{p+1} = (-1)^{n(p+1)+1} * d_p * \Lambda_{p+1} T^* \rightarrow \Lambda^p T^*.$$

We will suppress the subscripts from now on.

Let $\{dx^1, dx^2, dx^3\}$ be an orthonormal local basis and $w = f^1 dx^1 + f^2 dx^2 + f^3 dx^3$ a 1-form.

Then we have

$$\begin{aligned} dw = & \left(\frac{\partial f^2}{\partial x^1} - \frac{\partial f^1}{\partial x^2}\right) dx^1 \wedge dx^2 \\ & + \left(\frac{\partial f^3}{\partial x^2} - \frac{\partial f^2}{\partial x^3}\right) dx^2 \wedge dx^3 \\ & + \left(\frac{\partial f^1}{\partial x^3} - \frac{\partial f^3}{\partial x^1}\right) dx^3 \wedge dx^1 \end{aligned}$$

and

$$\delta w = (-1) \left(\frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3.$$

Thus if V is a vector field on $\Omega \subset R^3$ and w is the 1-form associated to V by means of the Riemannian structure on Ω, we have

$$* \delta w = -\mathrm{Div}\ V,$$

and

$$* dw = \mathrm{Curl}\ V.$$

If $k_2 = k_3$, then we have

$$\begin{aligned} W(n) = & \int_\Omega k_1 (\mathrm{Div}\ n)^2 + k_2 \langle n, \mathrm{Curl}\ n\rangle^2 \\ & + k_3 |n \times \mathrm{Curl}\ n|^2 \\ = & \int_\Omega k_1 (\mathrm{Div}\ n)^2 + k_2 |\mathrm{Curl}\ n|^2. \end{aligned}$$

We now define the energy functional E on a 3-manifold M, identifying the tangent bundle with the cotangent one by means of the Riemannian struture on M, as

$$\begin{aligned} E(w) = & \int_M k_1 |- * \delta w|^2 + k_2 | * dw|^2 \\ = & \int_M k_1 |\delta w|^2 + k_2 |dw|^2. \end{aligned}$$

We say that w is a liquid crystal if w is a minimizer of E with constraint $|w| = 1$.

In the following we study the existence of a liquid crystal. Let $H^{1,2}(T^*M)$ be the Sobolev space of forms on the manifold M, with norm defined by

$$\| w \|^2 = \int_M |w|^2 + |\delta w|^2 + |dw|^2.$$

Theorem 1. There is a liquid crystal in $H^{1,2}(T^*M)$.

Proof. Let Σ be the subset $\{w \in H^{1,2}(T^*M) : |w| = 1 \text{ a.e.}\}$ of $H^{1,2}(T^*M)$, which is weakly closed. Since E is coercive and weakly lower semi-continuous on Σ with respect to $H^{1,2}(T^*M)$, using the elementary fact in the calculus of variations (Struwe, 1990) we infer that E attains its minimum in Σ.

Remark. We may ask what regularity properties a minimizer w possesses. It would be also interesting to know if there is a gap phenomena, i.e., whether the minimum value of E among the Sobolev space is strictly less than that of E among the class of smooth vector fields.

Reference

Struwe, M, 1990, Variational Methods, Springer-Verlag, Berlin.

〈國文抄錄〉

3차원 다양체상의 액정 범함수

3차원 다양체상에 액정 범함수를 정의하고 최소 에너지를 갖는 벡터장이 소볼레프공간에 존재한다는 것을 보인다.