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HARMONIC MAPS OF COMPLETE RIEMANNIAN MANIFOLDS

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1. Introduction

The theory of harmonic mappings of a Riemannian manifold into another has been initiated by J. Eells and J. H. Sampson([2]) and studied by many authors. In particular, R. M. Schoen and S. T. Yau([3]) proved the following theorem:

Theorem A. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let N be a compact Riemannian manifold of nonpositive sectional curvature. Then every harmonic map of finite energy from M to N is constant.*

In this paper, we extend Theorem A under weaker assumptions by using Kato's inequality([1]) and characterize a harmonic map on complete Riemannian manifolds.

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2. Preliminaries

Let $\pi : E \to M$ be a Riemannian vector bundle over an m-dimensional manifold M, i.e., E is a vector bundle over M equipped with a C^∞-assignment of an inner product $<\ ,\ >$ to each fiber E_x of E over $x \in M$. Assume that a metric connection D is given on E, i.e., $D : A^p(E) \to A^{p+1}(E)$ is an $\mathbb{R}$-linear map such that if $f \in A^0$, $D(fs) = fDs + sdf$ and

$$d < s,t >=< Ds,t > + < s, Dt > \tag{2.1}$$

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for any $s,\ t \in A^p(E)$, where $A^p(E)$ is the set of all E-valued p-forms on M. Equivalently, for any $X \in TM$,

$$X < s,t > = < D_X s, t > + < s, D_X t > . \tag{2.2}$$

The Laplacian for $A^*(E)$ is by defition the operator

$$\Delta = DD^* + D^*D, \tag{2.3}$$

where D^* is the formal adjoint of D. Let $\{V_1, \cdots, V_m\}$ and $\{\omega^1, \cdots, \omega^m\}$ be a locally defined frame field and its dual coframe field respectively. Then on $A^*(E)$, we have

$$D = \sum_{i=1}^{m} \omega^i \wedge D_{V_i}, \quad D^* = -\sum_{j=1}^{m} i(V_j) D_{V_j}, \tag{2.4}$$

where $i(X)$ denotes the interior product operator with respect to X. Then on $A^*(E)$, we have the Weitzenböck formula

$$\Delta = -\sum_i D^2_{V_i V_i} + \sum_{k,\ell} \omega^k \wedge i(V_\ell) R_{V_k V_\ell}, \tag{2.5}$$

where $D^2_{XY} = D_X D_Y - D_{\nabla^M_X Y}$ and $R_{XY} = -[D_X, D_Y] + D_{[X,Y]}$ is the curvature tensor on $A^*(E)$. From this equation, we have

$$-\Delta^M |\Phi|^2 = 2\sum_i |D_{V_i}\Phi|^2 + 2 < \Phi, \sum_i D^2_{V_i V_i}\Phi >, \tag{2.6}$$

where$<\ ,\ >$ refers to the pointwise inner product, $|\Phi|^2 =< \Phi, \Phi >$, and $\Delta^M \equiv d\delta + \delta d$ is the ordinary Hodge Laplacian. From (2.5) and (2.6), for any harmonic section $\Phi \in A^*(E)$, we have

$$-\Delta^M |\Phi|^2 = 2\sum_i |D_{V_i}\Phi|^2 + 2 < \Phi, \sum_{k,\ell} \omega^k \wedge i(V_\ell) R_{V_k V_\ell}\Phi > . \tag{2.7}$$

In particular, let $\Phi = \sum_a e_a \phi^a \in A^1(E)$, where $\{e_1, \cdots, e_r\}$ is a local frame field of E and $\{\phi^a\}$ are 1-forms on M. Then we obtain

$$\begin{aligned} < \sum_{k,\ell} \omega^k \wedge i(V_\ell) R_{V_k V_\ell}\Phi, \Phi > &= \sum_{a,b,k,\ell} < R^E_{V_k V_\ell} e_a, e_b >< \omega^k \wedge i(V_\ell)\phi^a, \phi^b > \\ &+ \sum_{a,k,\ell} < \omega^k \wedge i(V_\ell) R^M_{V_k V_\ell}\phi^a, \phi^a >, \end{aligned}$$

where R^E and R^M are the curvature tensors on $\Gamma(E)$ and M, respectively. The second summand is equal to $\sum_a Ric^M(\phi_a, \phi_a)$, where Ric^M denotes the Ricci curvature tensor of M and ϕ_a stands for the vector field dual to ϕ^a. For the first summand, if we set $\phi^a = \sum \alpha_i^a \omega^i$, then $< \omega^i \wedge i(V_j)\phi^a, \phi^b >= \alpha_j^a \alpha_i^b$, so that

$$\sum_{a,b,k,\ell} < R^E_{V_k V_\ell} e_a, e_b >< \omega^k \wedge i(V_\ell)\phi^a, \phi^b > = \sum_{a,b,k,\ell} < R^E_{V_k V_\ell} e_a \alpha_\ell^a, e_b \alpha_k^b >$$
$$= \sum_{k,\ell} < R^E_{V_k V_\ell} \Phi(V_\ell), \Phi(V_k) > .$$

Hence for a harmonic E-valued 1-form Φ, we have

$$\begin{aligned} -\Delta^M |\Phi|^2 =& 2\sum_i |D_{V_i}\Phi|^2 + 2\sum_a Ric^M(\phi_a, \phi_a) \\ & - 2\sum_{i,j} < R^E_{V_i V_j} \Phi(V_i), \Phi(V_j) > . \end{aligned} \tag{2.8}$$

3. Harmonic maps

Let M and N be Riemannian manifolds with Riemannian metrics g and h respectively, and let their Levi-Civita connections be ∇^M and ∇^N respectively. Let $f : M \to N$ be a C^∞-map, and let $E \equiv f^*TN$ be the induced bundle over M. Then E has a naturally induced metric connection which we denote by $D \equiv f^*\nabla^N$. Also, the differential df of f gives naturally a cross section of the vector bundle $\mathrm{Hom}(TM, E)$ over M. Since $\mathrm{Hom}(TM, E)$ is canonically identified with $E \otimes T^*M (T^*M =$ the cotangent bundle of M), we see that df may be regarded as an E-valued 1-form, i.e., $df \in \Gamma(E \otimes T^*M) = A^1(E)$. Then $E \otimes T^*M$ has a naturally induced metric connection which we denote also by D. It is well known that $D(df)=0$([2]). Hence we say that the map f is *harmonic* if $D^*(df) = 0$. Also, f is said to be *totally geodesic* if for any $X \in TM$, $D_X df = 0$. It follows that totally geodesic maps are harmonic but the converse is not true in general. Let R^E be the curvature tensor of D. Then R^E is related to the curvature tensor R^N of ∇^N in the following way: let X, $Y \in T_xM$ and $s \in \Gamma(E)$, then

$$R^E_{XY} s = R^N_{df_x(X) df_x(Y)} s. \tag{3.1}$$

When a function ρ, local or global, is given on N, we shall identify it throughout this paper with the function $\rho \circ f$ induced on M.

Let $\{\bar{V}_a\}$ and $\{V_i\}$ be local frame fields on N and M respectively, and let $\{\bar{\omega}^a\}$ and $\{\omega^i\}$ be the their dual coframe fields respectively. Let $\phi^a \equiv f^*\bar{\omega}^a$. Then we can write df as $df = \sum_a \bar{V}_a \otimes \phi^a$. Hence if we set $\Phi = df$ in (2.8), then from (3.1) we obtain

$$
\begin{aligned}
-\Delta^M |df|^2 =& 2\sum_i |D_{V_i} df|^2 + 2\sum_a Ric^M(\phi_a, \phi_a) \\
& - 2\sum_{i,j} < R^N_{df(V_i)df(V_j)} df(V_i), df(V_j) >,
\end{aligned} \tag{3.2}
$$

where ϕ_a is the vector field dual to ϕ^a.

4. Main Theorems

Let $\rho(x)$ denote the least eigenvalue of $\mathcal{R}_x \equiv \sum_{k,\ell} \omega^k \wedge i(V_\ell) R_{V_k V_\ell}$ in E_x, where $\{V_j\}$ is a local frame field and $\{\omega^j\}$ is the dual coframe field of $\{V_j\}$. To prove our theorem, we prepare the following Kato's inequality:

Lemma 4.1([1]). *If $s \in \Gamma(E)$ satisfies $\Delta s = 0$, then*

$$\Delta^M |s| \leq -\rho |s|.$$

Let λ_0 is the infimum of the spectrum of the Laplacian Δ^M on L^2-functions on M. The proof of the following Theorem 4.2 is based on a method of P. Bérard ([1]).

Theorem 4.2. *Let M be a complete noncompact Riemannian manifold. If $\rho \geq -\lambda_0$ at all $x \in M$ and $\rho > -\lambda_0$ at some point x_0, then every harmonic map $f : M \to N$ of finite energy,i.e., $df \in L^2(M)$, is constant.*

proof. Since $\Delta df = 0$, by Lemma 4.1, we have

$$\Delta^M |df| \leq -\rho |df|.$$

By the assumption, we obtain

$$\Delta^M |df| \leq -\rho |df| \leq \lambda_0 |df|. \tag{4.1}$$

Since M is complete, one can construct function ω_ℓ such that $\omega_\ell \in C_0^\infty(M)$ and $\omega_\ell \equiv 1$ on $B(x_0, \ell)$, supp$\omega_\ell \subset B(x_0, 2\ell)$ and $|d\omega_\ell| \leq C/\ell$ for some constant C, where $\ell \in \mathbb{R}_+$, $x_0 \in M$ and $B(x_0, \ell)$ is the Riemannian open ball with radius ℓ and center x_0. Put $\varphi \equiv |df|$. Then from (4.1), we have

$$\int_M < d\omega_\ell^2\varphi, d\varphi > \leq -\int_M \omega_\ell^2 \rho\varphi^2 \leq \lambda_0 \int_M \omega_\ell^2\varphi^2. \tag{4.2}$$

Since $d\omega_\ell^2\varphi = 2\omega_\ell d\omega_\ell\varphi + \omega_\ell^2 d\varphi$, the left hand side of (4.2) can be written as

$$\int_M 2 < d\omega_\ell\varphi, \omega_\ell d\varphi > + \int_M \omega_\ell^2|d\varphi|^2 = \int_M |d(\omega_\ell\varphi)|^2 - \int_M \varphi^2|d\omega_\ell|^2. \tag{4.3}$$

From (4.2) and (4.3), we then obtain the inequality

(4.4)

$$\int_M |d(\omega_\ell\varphi)|^2 \leq \int_M \varphi^2|d\omega_\ell|^2 - \int_M \rho\omega_\ell^2\varphi^2 \leq \int_M \varphi^2|d\omega_\ell|^2 + \lambda_0 \int_M (\omega_\ell\varphi)^2.$$

Since λ_0 is the infimum of the spectrum of the Laplacian Δ^M acting on functions, we have

$$\int_M |d(\omega_\ell\varphi)|^2 \geq \lambda_0 \int_M (\omega_\ell\varphi)^2. \tag{4.5}$$

From (4.4) and (4.5), we have

$$\lambda_0 \int_M (\omega_\ell\varphi)^2 \leq \int_M \varphi^2|d\omega_\ell|^2 - \int_M \rho\omega_\ell^2\varphi^2 \leq \int_M \varphi^2|d\omega_\ell|^2 + \lambda_0 \int_M (\omega_\ell\varphi)^2.$$

Now, letting $\ell \to \infty$, we have

$$\lambda_0 \int_M \varphi^2 \leq -\int_M \rho\varphi^2 \leq \lambda_0 \int_M \varphi^2.$$

This implies that $\varphi = |df| = 0$ under our assumptions. Hence f is constant. □

Corollary 4.3. *Let M be a complete Riemannian manifold and let N be a Riemannian manifold of nonpositive sectional curvature. If $Ric^M \geq -\lambda_0$ at all $x \in M$ and $Ric^M > -\lambda_0$ at some point x_0, then every harmonic map $f : M \to N$ of finite energy, i.e., $df \in L^2(M)$, is constant.*

Proof. First, we recall the first Kato's inequality([1]): for any $s \in \Gamma(E)$, $|d|s|| \leq |Ds|$. It follows from the first Kato's inequality that if $\Delta s = 0$, then

$|s|\Delta^M|s| \le -< \mathcal{R}(s), s >$(cf.[1], p.263). Since $\Delta df = 0$ for a harmonic map $f : M \to N$, we get from the first Kato's inequality and (3.2)

$$|df|\Delta^M|df| \le -< \mathcal{R}(df), df >$$
$$= -\sum_a Ric^M(f^*\bar{w}^a, f^*\bar{w}^a) + \sum_{i,j} < R^N_{df(V_i)df(V_j)} df(V_i), df(V_j) > .$$

Note that if N has nonpositive sectional curvature, then we get

$$\sum_{i,j} < R^N_{df(V_i)df(V_j)} df(V_i), df(V_j) > \le 0.$$

Hence, if N has nonpositive sectional curvature, then we have

$$|df|\Delta^M|df| \le -\sum_a Ric^M(f^*\bar{w}^a, f^*\bar{w}^a).$$

If we assume $Ric^M \ge -\lambda_0$, then we have

$$|df|\Delta^M|df| \le \lambda_0|df|^2.$$

Hence we have $\Delta^M|df| \le \lambda_0|df|$. Then the rest of the proof goes as in the proof of Theorem 4.2. □

Now we are going to prove a characterization of harmonic maps on complete Riemannian manifolds. Take coordinate neiborhoods $\{U; x^j\}$ of M and $\{V, y^a\}$ of N in such a way that $f(U) \subset V$, where $(x^j) = (x^1, \cdots, x^m)$ and $(y^a) = (y^1, \cdots, y^n)$ are local coordinates of M and N respectively. The indices i, j, k run over the range $\{i, \cdots, m\}$ and the indices a, b, c the range $\{1, \cdots, m\}$. The summation convension will be used with respect to these two systems of indices. Suppose that $f : M \to N$ is respresented by equation $y^a = y^a(x^1, \cdots, x^m)$ with respect to $\{U; x^j\}$ and $\{V; y^a\}$. We put $A^a_i = \partial_i y^a(x^1, \cdots, x^m)$, where $\partial_i = \partial/\partial x^i$. Then the differential df of the mapping f is represented by the matrix (A^a_i) with respect to the local coordinates (x^j) and (y^a) of M and N. Let $X = X^j\partial_j$ and $Y = Y^j\partial_j$ are vector fields on M. If we put in U

$$A^a_{ji} = \nabla_j A^a_i,$$
$$\nabla_j A^a_i = \partial_j A^a_i + \Gamma^a_{bc} A^b_j A^a_i - \Gamma^k_{ji} A^a_k,$$

where Γ^a_{bc} and Γ^k_{ji} are Christoffel symbols with respect to ∇^N and ∇^M respectively, then $(A^a_{ji} X^j Y^i)\partial_a$ is the local expression of a vector field B defined along $f(M)$ and $A^a_{ji} = A^a_{ij}$. We now put

$$A^a = g^{ij} A^a_{ij},$$

where $(g^{ij}) = (g_{ij})^{-1}$. Then the vector field T with component A^a defined along $f(M)$ is called the *tension field* of $f : M \to N$. It is well known that $f : M \to N$ is harmonic if and only if $T = 0$([2]). If we put $\eta = (A_i^c A^b g_{bc})dx^i$, then η is a 1-form in M. Hence $*(\omega_\ell^2 \eta)$ is an (m-1)-form with compact support in $B(2\ell)$, where $*$ is the Hodge star operator. Then by Stokes' theorem, we have

$$\int_M d(*(\omega_\ell^2 \eta)) = 0.$$

Since $d(*(\omega_\ell^2\eta)) = -*\delta(\omega_\ell^2\eta)$, we have

$$\int_M *\delta(\omega_\ell^2\eta) = \int_{B(2\ell)} *\delta(\omega_\ell^2\eta) = 0.$$

Moreover, we have

$$\delta(\omega_\ell^2\eta) = \omega_\ell^2 \delta\eta - *(2\omega_\ell d\omega_\ell \wedge *\eta)$$

on M. Also, for a 1-form η, $\delta\eta = -\nabla^i \eta_i$. Hence we have

$$\begin{aligned}
\delta(\omega_\ell^2\eta) &= \omega_\ell^2\{-\nabla^i(A_i^c A^b g_{bc})\} - *(2\omega_\ell d\omega_\ell \wedge *\eta)\\
&= -\omega_\ell^2\{(\nabla^i A_i^c)A^b g_{bc} + A_i^c \nabla^i A^b g_{bc}\} - *(2\omega_\ell d\omega_\ell \wedge *\eta)\\
&= -\omega_\ell^2\{A^c A_b g_{bc}\} - \omega_\ell^2 A_i^c \nabla^i A^b g_{bc} - *(2\omega_\ell d\omega_\ell \wedge *\eta)\\
&= -|\omega_\ell T|^2 - < \omega_\ell df, \omega_\ell \nabla T > - * (2\omega_\ell d\omega_\ell \wedge *\eta).
\end{aligned}$$

Since $*(d\omega_\ell \wedge *\eta) = (d\omega_\ell)_i \eta^i = < d\omega_\ell, \eta >$, we have

$$\delta(\omega_\ell^2\eta) = -|\omega_\ell T|^2 - < \omega_\ell df, \omega_\ell \nabla T > -2 < d\omega_\ell, \omega_\ell \eta > .$$

By integration, we have

$$0 = \|\omega_\ell T\|^2_{B(2\ell)} + \ll \omega_\ell df, \omega_\ell \nabla T \gg_{B(2\ell)} + 2 \ll d\omega_\ell, \omega_\ell \eta \gg_{B(2\ell)} .$$

By Schwartz's inequality and $|d\omega_\ell| \le C/\ell$, we get

$$0 \ge \|\omega_\ell T\|^2_{B(2\ell)} - \|\omega_\ell df\| \|\omega_\ell \nabla T\|_{B(2\ell)} - \frac{C}{\ell}\|\omega_\ell \eta\|_{B(2\ell)}.$$

For any finite energy map f, i.e., $|df| \in L^2(M)$, letting $\ell \to \infty$, we have

$$0 \ge \|T\|^2 - \|df\|\|\nabla T\|.$$

From this inequality, we have the following:

Theorem 4.4. *Let M be a complete Riemannian manifold. If a mapping $f : M \to N$ has a finite energy and $\nabla T = 0$, then f is harmonic.*

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