Far East J. Math. Sci. (FJMS)
Special Volume (2000), Part II
(*Geometry* and *Topology*), 169-177

TRANSVERSAL INFINITESIMAL AUTOMORPHISMS FOR NON-HARMONIC KÄHLER FOLIATION

SEOUNG DAL JUNG

(Received February 7, 2000)

Submitted by K. K. Azad

Abstract

We study transversally infinitesimal automorphisms and some vanishing theorems on non-harmonic Kähler foliation.

1. Introduction

A foliation $\mathcal{F}$ on a manifold M is *harmonic* if the canonical projection $\pi : TM \to Q$ of the tangent bundle to the normal bundle Q is a harmonic Q-valued 1-form. The foliation $\mathcal{F}$ is harmonic if and only if its tension field τ is zero. In 1988, Nishikawa and Tondeur [4] introduced the Kähler foliation and proved the following theorem:

Theorem A. *Let $\mathcal{F}$ be a harmonic Kähler foliation on a closed orientable manifold M with transversal Ricci operator $\rho_\nabla \leq 0$. Then every transversally holomorphic infinitesimal automorphism $Y \in V(\mathcal{F})$ satisfies $\nabla\pi(Y) = 0$. If $\rho_\nabla < 0$ at some point $x \in M$, then every $Y \in V(\mathcal{F})$ with transversally holomorphic s satisfies $Y \in \Gamma L$.*

In this paper, we study the transversally holomorphic infinitesimal

2000 Mathematics Subject Classification: 53C12, 57R30.

Key words and phrases: Kähler foliation, infinitesimal automorphism.

automorphisms on non-harmonic Kähler foliation. Also, we study the relation between Killing fields and transverse Killing fields on M. Generally, a Killing field of M is a transversally Killing field. But the converse is not true. We give some conditions under which the converse holds. Throughout this paper, we have the following indices:

$$1 \le i, j, \cdots \le p; \quad 1 \le a, b, \cdots \le n,$$

$$1 \le \alpha, \beta, \cdots \le q(= 2n), \quad 1 \le A, B, \cdots \le p + q.$$

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional compact Riemannian manifold with a Kähler foliation $\mathcal{F}$ [4] of codimension $q(= 2n)$. Then there exists an exact sequence of vector bundles

$$0 \to L \to TM \overset{\pi}{\to} Q \to 0, \tag{2.1}$$

where L is the tangent bundle and Q is the normal bundle of $\mathcal{F}$ with respect to g_M. Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying

$$[Y, Z] \in \Gamma(L) \tag{2.2}$$

for any $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$. We set

$$\Gamma Q^L = \{s \in \Gamma Q \mid s = \pi(Y), Y \in V(\mathcal{F})\}. \tag{2.3}$$

Then $s \in \Gamma Q^L$ satisfies $\nabla_X s = 0$ for any $X \in \Gamma L$ [2]. From (2.1), we have associated exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathcal{F}) \overset{\pi}{\to} \Gamma Q^L \to 0. \tag{2.4}$$

Let $J : Q \to Q$ be the holonomy invariant almost complex structure on Q and let the holonomy invariant metric g_Q be Hermitian, i.e., $g_Q(JX, JY) = g_Q(X, Y)$ for $X, Y \in \Gamma Q$. A torsion free and metric connection ∇ in Q is defined by

$$\nabla_X s = \pi([X, Y_s]) \qquad \text{for } X \in \Gamma(L),$$

$$\nabla_X s = \pi(\nabla^M_X Y_s) \qquad \text{for } X \in \Gamma(Q), \tag{2.5}$$

where $s \in \Gamma(Q)$, $\pi(Y_s) = s$ and ∇^M is a Levi-Civita connection on M. The curvature R_∇ of ∇ is defined by

$$R_\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s$$

for any $X, Y \in \Gamma TM$ and $s \in \Gamma(Q)$. Then we have the following identities [4]:

$$R_\nabla(X, Y)J = JR_\nabla(X, Y),$$

$$R_\nabla(JX, JY) = R_\nabla(X, Y),$$

$$R_\nabla(X, Y)Z + R_\nabla(Y, Z)X + R_\nabla(Z, X)Y = 0 \tag{2.6}$$

for elements X, Y and Z of ΓQ. Let $\{e_A\}$ be an oriented orthonormal basis of $T_x M$ with e_i in L_x and e_α in $L_x^\perp$ ($\mathcal{F}$ is of codimension $q = 2n$ on M^{p+q}). The transversal Kähler property of $\mathcal{F}$ allows then to extend e_a, Je_a to local vector fields E_a, $JE_a \in \Gamma L^\perp$ such that

$$(\nabla_{E_a} E_b)_x = (\nabla_{E_a} JE_b)_x = (\nabla_{JE_a} E_b)_x = (\nabla_{JE_a} JE_b)_x = 0. \tag{2.7}$$

As a consequence of torsion freeness [1]

$$[E_a, E_b]_x, \quad [E_a, JE_b]_x, \quad [JE_a, JE_b]_x \in L_x. \tag{2.8}$$

Then E_a, JE_a can be chosen as (local) infinitesimal automorphisms of $\mathcal{F}$, so that

$$\nabla_X E_\alpha = \pi[X, E_\alpha] = 0 \qquad \text{for } X \in \Gamma L. \tag{2.9}$$

We can complete E_a, JE_a by the Gram-Schmidt process to a moving local frame by adding $E_i \in \Gamma L$ with $(E_i)_x = e_i$. In terms of such a moving frame the transversal Ricci operator $\rho_\nabla : Q \to Q$ is given by [4]

$$\rho_\nabla = \sum JR_\nabla(E_a, JE_a). \tag{2.10}$$

3. Infinitesimal Automorphisms

The transverse Lie derivative $\theta(Y)$ with respect to $Y \in V(\mathcal{F})$ is defined by

$$\theta(Y)s = \pi([Y, Y_s]) \tag{3.1}$$

for any $s \in \Gamma Q$ with $\pi(Y_s) = s$.

Definition 3.1. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 0$, then $s = \pi(Y)$ is called a *transverse Killing field* of $\mathcal{F}$.

If g_M is a bundle-like metric on M and $Y \in \Gamma TM$ is a Killing vector field for (M, g_M), then $\pi(Y)$ is transversal Killing field for g_Q. But the converse is not necessarily true: $Y \in V(\mathcal{F})$ may satisfy $\theta(Y)g_Q = 0$ without satisfying $\theta(Y)g_M = 0$. Under what condition, does the converse hold? For any $Y \in V(\mathcal{F})$, we have

$$(\theta(Y)g_M)(Z, W) = Yg_M(Z, W) - g_M(\theta(Y)Z, W) - g_M(Z, \theta(Y)W)$$

for any $Z, W \in \Gamma TM$. From (3.1), we get

$$g_M(\theta(Y)Z, W) = g_M(\theta(Y)\pi(Z), \pi(W)) + g_M(\theta(Y)\pi^{\perp}(Z), \pi^{\perp}(W)).$$

Hence we have

$$(\theta(Y)g_M)(Z, W) = (\theta(Y)g_L)(\pi^{\perp}(Z), \pi^{\perp}(W)) + \big(\theta(Y)g_Q\big)(\pi(Z), \pi(W)).$$

From this equality, we have

Proposition 3.2. *Let $\pi(Y)$ be the transversal Killing field of M. Then we have*

$$(\theta(Y)g_M)(Z, W) = (\theta(Y)g_L)(\pi^{\perp}(Z), \pi^{\perp}(W))$$

for any $Z, W \in \Gamma TM$.

From Proposition 3.2, we have

Corollary 3.3. *Let $\pi(Y)$ be the transversal Killing field of M. Then Y is a Killing field of M if and only if $\theta(Y)g_L = 0$.*

By straight calculation, we have

$$(\theta(Y)\, g_L)(\pi^{\perp}(Z), \pi^{\perp}(W)) = g_M\left(\nabla^M_{\pi^{\perp}(Z)} Y, \pi^{\perp}(W)\right) + g_M\left(\pi^{\perp}(Z), \nabla^M_{\pi^{\perp}(W)} Y\right).$$

From this equality, we get

Corollary 3.4. *Let $\pi(Y)$ be the transversal Killing field of M. If the infinitesimal automorphism Y is parallel along the leaves, then Y is a Killing field of M.*

It is well known that $\mathcal{F}$ is totally geodesic if and only if

$$g_M(\nabla^M_Z \pi(Y), W) + g_M(Z, \nabla^M_W \pi(Y)) = 0$$

for $Z, W \in \Gamma L$ and $Y \in V(\mathcal{F})$ [9]. Hence we have

Corollary 3.5. *Let $\mathcal{F}$ be the totally geodesic foliation. Let $\pi(Y)$ be the transversal Killing field of M. If the tangential part of the infinitesimal automorphism Y is parallel along the leaves, then Y is a Killing field of M.*

Definition 3.6. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)J = 0$, then $s = \pi(Y)$ is called a *transversally holomorphic*.

By definition, for $Z \in \Gamma L^{\perp}$

$$(\theta(Y)\, J)(Z) = \theta(Y)(JZ) - J(\theta(Y)\, Z).$$

But the second part of this equation equals $\pi[Y, JZ] - J\pi[Y, Z]$, which yields the formula

$$(\theta(Y)\, J)(Z) = -\nabla_{JZ} s + J\nabla_Z s$$

for $Y \in V(\mathcal{F})$ and $s = \pi(Y)$. Hence $s = \pi(Y)$ is a transversally holomorphic if and only if

$$\nabla_{JZ} s = J\nabla_Z s \tag{3.2}$$

for all $Z \in \Gamma L^{\perp}$.

4. Vanishing Theorems

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. Let

$d_\nabla : \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$ be the exterior differential operator and the operator $d_\nabla^* : \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)$ is the adjoint operator of d_∇ with respect to $<< \, , >>$, where

$$<< \eta, \eta' >> = \int_M g_Q(\eta \wedge *\eta')$$

for $\eta, \eta' \in \Omega^r(M, Q)$. The Laplacian Δ acting on $\Omega^r(M, Q)$ is defined by

$$\Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla .$$

Definition 4.1. The transversal Jacobi operator of Riemannian foliation is given by

$$J_\nabla s = (\Delta - \rho_\nabla - \nabla_\tau) s$$

for $s \in \Gamma Q$. Then $s \in Ker\, J_\nabla$ is called a *transversal Jacobi field* of $\mathcal{F}$.

We calculate the Laplacian of $s = \pi(Y)$:

$$(\Delta s)_x = (d_\nabla^* d_\nabla s)_x = -\sum_{A=1}^{m} (\nabla_{e_A}(d_\nabla s))(e_A) = -\sum_{A=1}^{m} \left(\nabla_{e_A} \nabla_{E_A} s - \nabla_{\nabla^M_{e_A} E_A} s \right).$$

Since $\nabla_{e_a} E_a = \pi(\nabla^M_{e_a} E_a) = 0$, we have $\nabla^M_{e_a} E_a \in L_x$. This implies that $\nabla_{\nabla^M_{e_A} E_A} s = \nabla_{\tau_x} s$, where $\tau_x = \sum_{i=1}^{p} \pi(\nabla^M_{e_i} E_i)$ is the mean curvature vector field of F. Hence we have

$$\Delta s = -\sum_{a=p+1}^{p+n} \nabla_{e_a} \nabla_{E_a} s - \sum_{a=p+1}^{p+n} \nabla_{Je_a} \nabla_{JE_a} s + \nabla_\tau s. \qquad (4.1)$$

Lemma 4.2 [5]. *If $s = \pi(Y)$ is a transversally holomorphic, then s is a transversal Jacobi field of $\mathcal{F}$, i.e.,*

$$\Delta s = \rho_\nabla s + \nabla_\tau s .$$

Proof. Since $s = \pi(Y)$ projectable, (2.8) implies that $\nabla_{[Je_a, e_a]} s = 0$. By (3.2), we have

$$R_{\nabla}(Je_a, e_a)s = J\nabla_{e_a}\nabla_{E_a}s + J\nabla_{Je_a}\nabla_{JE_a}s.$$

Therefore, we get

$$-\sum_a \nabla_{e_a}\nabla_{E_a}s - \sum_a \nabla_{Je_a}\nabla_{JE_a}s = \sum_a JR_{\nabla}(Je_a, e_a)s = \rho_{\nabla}s.$$

From (4.1), we obtain the result.

Theorem 4.3 [5]. *If a transversal Jacobi field* $s = \pi(Y)$ *satisfies* $g_Q((\theta(Y)J)J\tau, s) = 0$, *then* $s = \pi(Y)$ *is a transversally holomorphic.*

Next, we evaluate for $x \in M$

$$< \theta(Y)J, \theta(Y)J >_x$$

$$= \sum_a g_Q((\theta(Y)J)e_a, (\theta(Y)J)e_a) + g_Q((\theta(Y)J)Je_a, (\theta(Y)J)Je_a)$$

$$= 2\sum_a Je_a g_Q(\nabla_{Je_a}s - J\nabla_{e_a}s, s) + 2\sum_a e_a g_Q(\nabla_{Je_a}s + \nabla_{e_a}s, s)$$

$$- 2\sum_a g_Q(\nabla_{Je_a}\nabla_{JE_a}s + \nabla_{e_a}\nabla_{E_a}s, s) + 2\sum_a g_Q(\nabla_{Je_a}J(\nabla_{E_a}s))$$

$$-\nabla_{e_a}J(\nabla_{JE_a}s, s)$$

$$= 2div_B Z + g_Q(\Delta s - \nabla_\tau s, s) + 2g_Q(JR_{\nabla}(Je_a, e_a)s, s),$$

where $Z \in \Gamma Q$ is defined by

$$g_Q(Z, X) = g_Q(\nabla_X s + J\nabla_{JX}s, s) \quad \text{for} \quad X \in \Gamma Q$$

and the transversal divergence $div_B Z$ of Z is defined as the unique scalar satisfying $\theta(Z)\nu = div_B Z \cdot \nu$, ν being the transversal volume form defined by g_Q [3]. It follows that

$$< \theta(Y)J, \theta(Y)J >_x = 2(div_B Z)_x + 2g_Q(\Delta s, s)_x$$
$$-2g_Q(\rho_{\nabla}(s), s)_x - 2g_Q(\nabla_\tau s, s)_x . \tag{4.2}$$

By the transversal divergence theorem [10], we have

$$\int_M div_B Z = \ll Z,\ \tau \gg$$

$$= \ll \nabla_\tau s + J\nabla_{J\tau} s,\ s \gg .$$

Integrating (4.2), we have

Proposition 4.4. *For* $Y \in V(\mathcal{F})$, *we get*

$$\frac{1}{2} \ll \theta(Y)J,\ \theta(Y)J \gg = \ll \Delta s,\ s \gg - \ll \rho_\nabla s,\ s \gg + \ll J\nabla_{J\tau} s,\ s \gg ,$$

where $s = \pi(Y)$.

Let $s \in \Gamma Q^L$ be any section. Then we obtain the classical identity [2]

$$-\frac{1}{2}\Delta g_Q(s,\ s) = g_Q(\nabla s,\ \nabla s) - g_Q(\Delta s,\ s). \tag{4.3}$$

The Laplacian on the LHS is the ordinary Laplacian $d^* d$ of the function on M. Summing up Proposition 4.4 and (4.3), we have

Proposition 4.5. *For* $Y \in V(\mathcal{F})$ *and* $s = \pi(Y)$,

$$-\frac{1}{2}\int_M \Delta |\, s\,|^2 = -\frac{1}{2}\|\,\theta(Y)J\,\|^2 - \ll \rho_\nabla s,\ s \gg + \ll J\nabla_{J\tau} s,\ s \gg + \|\,\nabla s\,\|^2.$$

Corollary 4.6. *If* $Y \in V(\mathcal{F})$ *is a transversally holomorphic, then*

$$-\frac{1}{2}\int_M \Delta |\, s\,|^2 = - \ll \rho_\nabla s,\ s \gg - \ll \nabla_\tau s,\ s \gg + \|\,\nabla s\,\|^2.$$

From Corollary 4.6, we obtain

Theorem 4.7. *Let* $\mathcal{F}$ *be a Kähler foliation on a closed manifold M with transversal Ricci operator* $\rho_\nabla \le 0$. *If the transversally holomorphic infinitesimal vector field is parallel in direction to the mean curvature vector, then* $\nabla\pi(Y) = 0$. *Moreover, if* $\rho_\nabla < 0$ *at some point* $x \in M$ *and* $\nabla_\tau \pi(Y) = 0$, *then* $\pi(Y) = 0$, *i.e.,* $Y \in \Gamma L$.

References

[1] F. W. Kamber and Ph. Tondeur, Harmonic foliations, Proc. NSF Conference on Harmonic Maps, Springer Lecture Notes 949 (1982), 87-121.

[2] F. W. Kamber and Ph. Tondeur, Infinitesimal automorphisms and second variation of the energy for harmonic foliations, Tohoku Math. J. 34 (1982), 525-538.

[3] F. W. Kamber, Ph. Tondeur and G. Toth, Transversal Jacobi fields for harmonic foliations, Michigan Math. J. 34 (1987), 261-266.

[4] S. Nishikawa and Ph. Tondeur, Transversal infinitesimal automorphisms for harmonic Kähler foliations, Tohoku Math. J. 40 (1988), 599-611.

[5] S. Nishikawa and S. Yorozu, Transversal infinitesimal automorphisms for foliations, preprint.

[6] B. O'Neill, The fundamental equation of a submersion, Michigan Math. J. 13 (1966), 459-469.

[7] J. S. Pak and S. Yorozu, Transverse fields on foliated Riemannian manifolds, J. Korean Math. Soc. 25 (1988), 83-92.

[8] Ph. Tondeur and G. Toth, On transversal infinitesimal automorphisms for harmonic foliations, Geometriae Dedicata 24 (1987), 229-236.

[9] Ph. Tondeur, Foliations on Riemannian Manifolds, Springer-Verlag, New York-Berlin-London-Paris-Tokyo, 1988.

[10] S. Yorozu and T. Tanemura, Green's theorem on a foliated Riemannian manifold and its applications, Acta Math. Hungar. 56 (1990), 239-245.

Department of Mathematics
Cheju National University
Cheju 690-756, Korea