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ON THE SPECTRAL NORM AND EIGENVALUES OF W_c-COMPANION MATRICES

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Abstract: An $n \times n$ matrix A is called W_c-companion matrix if A is defined by

$$A = \begin{bmatrix} 0 & W_c \\ a_0 & \mathbf{d}^* \end{bmatrix}$$

with $a_0 \neq 0$, $\mathbf{d}^* = (a_1, \ldots, a_{n-1})$ and $W_c W_c^* = W_c^* W_c = cI_{n-1}$ for some positive real number c. A companion matrix is a special case of W_c-companion matrix. We give the explicit spectral norm and eigenvalue bounds for W_c-companion matrix.

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1. Introduction

We consider a monic complex polynomial

$$p(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0.$$

Is there a square matrix A whose minimal polynomial is $p(z)$? If so, the size of the matrix A must be at least n; it is not hard to find such a matrix $A \in M_n(\mathbf{C})$ when $\mathbf{C}$ is a complex field. Let I_k be the identity matrix of order k. Consider the matrix

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$$C = \begin{bmatrix} 0 & I_{n-1} \\ a_0 & \mathbf{d}^* \end{bmatrix},$$

with $a_0 \neq 0$ and $\mathbf{d}^* = \bar{\mathbf{d}}^T = (a_1, a_2, \ldots, a_{n-1})$. The matrix C is known as the $n \times n$ nonsingular companion matrix of the polynomial $p(z)$.

The characteristic polynomial of the companion matrix C is $p(z)$. If $\tilde{z}$ is a root of $p(z) = 0$ and if $|||\cdot|||$ is any matrix norm on M_n, then $|\tilde{z}| \leq |||C|||$. We have very famous bounds as following:

$$\begin{aligned} |\tilde{z}| &\leq \max\{|a_0|, 1 + |a_1|, \ldots, 1 + |a_{n-1}|\} \\ &\leq 1 + \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}, \end{aligned}$$

this bound on the roots is known as Cauchy's bound, and

$$\begin{aligned} |\tilde{z}| &\leq \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} \\ &\leq 1 + |a_0| + |a_1| + \cdots + |a_{n-1}| \end{aligned}$$

which is known as Montel's bound.

Now, our natural question becomes, what are the roots of the characteristic polynomial of

$$\tilde{C} = \begin{bmatrix} 0 & W \\ a_0 & \mathbf{d}^* \end{bmatrix}$$

for a permutation matrix W, for an orthogonal matrix W, for a unitary matrix W or, more general, for $W_c W_c^* = W_c^* W_c = cI_{n-1}$?

For a positive real number c, let W_c be an $(n-1) \times (n-1)$ complex matrix satisfying $W_c W_c^* = W_c^* W_c = cI_{n-1}$. Now, we introduce the concept of a W_c-companion matrix. The $n \times n$ matrix A is called *W_c-companion matrix* if A is defined by

$$A = \begin{bmatrix} 0 & W_c \\ a_0 & \mathbf{d}^* \end{bmatrix} \tag{1.1}$$

with $a_0 \neq 0$, $\mathbf{d}^* = (a_1, \cdots, a_{n-1})$. If W_c is the identity matrix, then A is the companion matrix C of polynomial $p(z)$. So, the matrix A is a generalization of the companion matrix C. Let $\mathcal{A}_c$ be the set of all W_c-companion matrix A for a positive real number c.

Let B be an $m \times n$ matrix. The *singular values $\sigma_i(B)$* of B are nonnegative square roots of the eigenvalues, $\lambda_i(BB^*)$, of the positive semidefinite matrix BB^*, or equivalently, they are the eigenvalues of the positive semidefinite square root $(BB^*)^{\frac{1}{2}}$, so that

$$\sigma_i(B) = [\lambda_i(BB^*)]^{\frac{1}{2}} = \lambda_i[(BB^*)^{\frac{1}{2}}], \quad i = 1, 2, \cdots, m.$$

The singular values are real and nonnegative. The vector of singular values of B arranged in nonincreasing order is denoted by $\sigma(B) = (\sigma_1(B), \cdots, \sigma_m(B))$. For a positive semidefinite Hermitian matrix B, $\sigma_i(B) = \lambda_i(B), i = 1, 2, \cdots, n$.

Unfortunately, in general, a W_c-companion matrix A is not positive semidefinite Hermitian.

In Section 2, we introduce explicit spectral norm and eigenvalue bounds of W_c-companion matrix. In Section 3, we give an explicit spectral norm of the following form of a matrix, G;

$$G = \begin{bmatrix} \mathbf{x} & I_{n-1} \\ a_0 & \mathbf{d}^* \end{bmatrix},$$

where $\mathbf{x} = (b, 0, \ldots, 0)^T$ and b is a complex number. If $b = 0$, then G is a companion matrix. So, we can say that the matrix G is another generalized companion matrix.

2. Spectral Norm of W_c-companion Matrices

Let B be an $n \times n$ matrix. The spectral norm of B is defined by

$$|||B|||_2 \equiv \max\{\sqrt{\lambda} \ : \ \lambda \text{ is an eigenvalue of } BB^*\},$$

where B^* is the conjugate transpose of B. Notice that if $B^*B\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^*B^*B\mathbf{x} = ||B\mathbf{x}||_2^2 = \lambda||\mathbf{x}||_2^2$, so $\lambda \geq 0$ and $\sqrt{\lambda}$ is nonnegative real number.

In this section, we give an explicit spectral norm and give bounds for eigenvalues of W_c-companion matrix in (1).

Let A be an $n \times n$ matrix in $\mathcal{A}_c$. Then

$$AA^* = \begin{bmatrix} cI_{n-1} & W_c\mathbf{d} \\ (W_c\mathbf{d})^* & s \end{bmatrix}, \tag{2.1}$$

where $s = \sum_{i=0}^{n-1} |a_i|^2 = |a_0|^2 + ||\mathbf{d}||^2$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of A and $\mathcal{D}$ be the set

$$\{(x_1, x_2, \ldots, x_n) \in R^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\},$$

and let

$$R_+^n = \{(x_1, \ldots, x_n) \in R^n | x_i > 0 \text{ for all } i\}.$$

Lemma 2.1. *Let A be an $n \times n$ matrix in $\mathcal{A}_c$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of A and $(\sigma_1^2, \ldots, \sigma_n^2) \in \mathcal{D}$. Then $\sigma_2^2 = \cdots = \sigma_{n-1}^2 = c$, and σ_1^2, σ_n^2 are roots of $f(z) = z^2 - (s+c)z + c|a_0|^2$.*

Proof. Since $A \in \mathcal{A}_c$, $W_c^* W_c = cI_{n-1}$ and $s - |a_0|^2 = ||\mathbf{d}||^2$, by Schur complements [2], we can easily verify that

$$\begin{aligned}\det(zI_n - A_cA_c^*) &= (z-c)^{n-1}[(z-s) - (-W_c\mathbf{d})^*((z-c)I_{n-1})^{-1}(-W_c\mathbf{d})] \\ &= (z-c)^{n-1}[(z-s) - \tfrac{1}{z-c}(W_c\mathbf{d})^*(W_c\mathbf{d})] \\ &= (z-c)^{n-1}[(z-s) - \tfrac{c}{z-c}||\mathbf{d}||^2] \\ &= (z-c)^{n-2}(z^2 - (s+c)z + c|a_0|^2).\end{aligned}$$

Thus AA^* has c as an eigenvalue of multiplicity at least $(n-2)$. Since the eigenvalues of the principal submatirx cI_{n-1} in (2.1) interlace with those of AA^*, it follows that $\sigma_n^2 \le c \le \sigma_1^2$ and σ_1^2, σ_n^2 are roots of $f(z)$.

Since σ_1^2 and σ_n^2 are roots of $f(z)$, we have $\sigma_1\sigma_n = \sqrt{c}|a_0|$ and $\sigma_1^2+\sigma_n^2 = s+c$. Since

$$||\mathbf{d}||^2 = s - \frac{\sigma_1^2\sigma_n^2}{c} = -\frac{1}{c}(\sigma_1^2 - c)(\sigma_n^2 - c),$$

we have $\sigma_1 + \sigma_n = \alpha$ and $||\mathbf{d}||^2 = (\alpha + |a_0| + \sqrt{c})(\alpha - |a_0| - \sqrt{c})$, where

$$\alpha = \sqrt{s + c + 2\sqrt{c}|a_0|}. \tag{2.2}$$

Set $\beta = \sqrt{s + c - 2\sqrt{c}|a_0|}$. Then the following theorem holds.

Theorem 2.2. *Let A be an $n \times n$ matrix in $\mathcal{A}_c$. Then*

$$|||A|||_2 = \sigma_1 = \frac{\alpha+\beta}{2},$$

where $\alpha = \sqrt{s + c + 2\sqrt{c}|a_0|}$ and $\beta = \sqrt{s + c - 2\sqrt{c}|a_0|}$.

Proof. By Lemma 2.1, the eigenvalues of AA^* are c and

$$\frac{(s+c) \pm \sqrt{(s+c)^2 - 4c|a_0|^2}}{2}.$$

Since $||\mathbf{d}||^2 \ge 0$, $\sigma_1 \ge \sqrt{c}$ and hence

$$|||A|||_2 = \sigma_1 = \sqrt{\frac{s+c+\sqrt{(s+c)^2 - 4c|a_0|^2}}{2}} = \sqrt{\frac{\alpha^2 + \beta^2 + 2\alpha\beta}{4}} = \frac{\alpha+\beta}{2},$$

where $\alpha = \sqrt{s + c + 2\sqrt{c}|a_0|}$ and $\beta = \sqrt{s + c - 2\sqrt{c}|a_0|}$.

Note that the spectral norm of a matrix A in $\mathcal{A}_1$ does not depend on W_1. Thus, we have the following corollary.

Corollary 2.3. *Let A be an $n \times n$ matrix in $\mathcal{A}_1$. Then*

$$|||A|||_2 = \frac{\sqrt{s+1+2|a_0|}+\sqrt{s+1-2|a_0|}}{2}.$$

Let $p(z)$ be the characteristic polynomial of A in $\mathcal{A}_c$. If $c \leq 1$, then we have the same inequalities that Cauchy's bound and Montel's bound of the companion matrix C.

In particular, if $a_0 = 1$ and $\mathbf{d}^* = (1, 1, \dots, 1)$, then

$$Q_n = \begin{bmatrix} 0 & I_{n-1} \\ 1 & \mathbf{d}^* \end{bmatrix}$$

is a W_1-companion matrix. The matrix Q_n is said to be *n-generalized Fibonacci matrix*. Since $Q_n \in \mathcal{A}_1$, from Corollary 2.3, we have

$$|||Q_n|||_2 = \frac{\sqrt{n+3}+\sqrt{n-1}}{2}.$$

In [3], we can find the following inequalities:

$$\sqrt{\varphi} \leq |\lambda_n(Q_n)| < 1 \quad \text{and} \quad \sqrt{w} \leq |\lambda_1(Q_n)| < 2,$$

where

$$\varphi = \frac{n+1-\sqrt{n^2+2n-3}}{2} \quad \text{and} \quad w = \frac{2n-1}{n}.$$

Let $p(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0$, $a_0 \neq 0$. If $\tilde{z}$ is a root of $p(z) = 0$ and if $|||\cdot|||$ is any matrix norm on M_n, then $|\tilde{z}| \leq |||C|||$, where C is an $n \times n$ companion matrix associated with $p(z)$. Let

$$q(z) = -\frac{1}{a_0} z^n p(\frac{1}{z}).$$

Then the roots of $q(z) = 0$ are exactly reciprocals of the roots of $p(z) = 0$. So, we have the following corollary which is about a lower bound of the roots of $p(z) = 0$.

Corollary 2.4. *Let $\tilde{z}$ be a root of the characteristic equation of a companion matrix C. Then*

$$|\tilde{z}| \geq \frac{\alpha' - \beta'}{2},$$

where $\alpha' = \sqrt{s+1+2|a_0|}$ and $\beta' = \sqrt{s+1-2|a_0|}$.

Proof. Since $q(z) = -\frac{1}{a_0} z^n p(\frac{1}{z})$ and using Theorem 2.2, we get

$$\left|\frac{1}{\bar{z}}\right| \leq \frac{\alpha' + \beta'}{2|a_0|}.$$

Since $(\alpha')^2 - (\beta')^2 = 4|a_0|$,

$$|\bar{z}| \geq \frac{2|a_0|(\alpha' - \beta')}{(\alpha' + \beta')(\alpha' - \beta')} \geq \frac{\alpha' - \beta'}{2}.$$

For $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n$$

and equality holds when $k = n$, then the vector $\mathbf{x}$ is said to be *majorized* by the vector $\mathbf{y}$ and denoted by $\mathbf{x} \prec \mathbf{y}$. For $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n,$$

$\mathbf{x}$ is said to be *submajorized* by $\mathbf{y}$ and denoted by $\mathbf{x} \prec_w \mathbf{y}$. The following fact is well known;

$$(\bar{x}, \ldots, \bar{x}) \prec (x_1, \ldots, x_n),$$

where $\bar{x} = (\sum_{i=1}^{n} x_i)/n$.

A real valued function ϕ defined on a set $F \subseteq R^n$ is said to be *Schur-convex* on F if $\mathbf{x} \prec \mathbf{y} \implies \phi(\mathbf{x}) \leq \phi(\mathbf{y})$. Similarly, ϕ is said to be *Schur-concave* on F if $\mathbf{x} \prec \mathbf{y} \implies \phi(\mathbf{x}) \geq \phi(\mathbf{y})$.

Lemma 2.5. *[4, 3.D.6] The function*

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \tilde{x}_i$$

is a Schur-convex on R_+^n, *where* $\tilde{x}_i = \dfrac{x_i}{x_1 \cdots x_{i-1} x_{i+1} \cdots x_n}$.

Lemma 2.6. *[4, 9.E.1.b] For any nonsingular complex matrix A of order n,*

$$(|\lambda_1(A)|^2, \cdots, |\lambda_n(A)|^2) \prec_w (\sigma_1^2(A), \cdots, \sigma_n^2(A)),$$

where $\lambda_i(A)$ are eigenvalues of A and $\sigma_i(A)$ are singular values of A, $i = 1, 2, \cdots, n$.

We now consider the eigenvalues of W_c-companion matrix A.

Lemma 2.7. *Let A be an $n \times n$ matrix in $\mathcal{A}_c$ and let $\lambda_1, \ldots, \lambda_n$ be ordered eigenvalues of A so that $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{D}$. Then*

$$\left(\frac{n|a_0|c^{\frac{n-1}{2}}}{(n-1)c+s}\right)^{\frac{1}{n-2}} \leq |\lambda_1|.$$

Proof. We know that $|\det A| = |a_0|c^{\frac{n-1}{2}}$ and $(\bar{\lambda}, \ldots, \bar{\lambda}) \prec (|\lambda_1|, \ldots, |\lambda_n|)$, where $\bar{\lambda} = (\sum_{i=1}^n |\lambda_i|)/n$.

By Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned} n\frac{\bar{\lambda}}{\bar{\lambda}^{n-1}} = \frac{n}{\bar{\lambda}^{n-2}} &\leq \frac{1}{|a_0|c^{\frac{n-1}{2}}}(|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_n|^2) \\ &\leq \frac{1}{|a_0|c^{\frac{n-1}{2}}}(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2). \end{aligned}$$

Since $\sum_{i=1}^n \sigma_i^2 = (n-1)c + s$, we have

$$\frac{n}{\bar{\lambda}^{n-2}} \leq \frac{1}{|a_0|c^{\frac{n-1}{2}}}((n-1)c+s).$$

So,

$$\frac{n|a_0|c^{\frac{n-1}{2}}}{(n-1)c+s} \leq \bar{\lambda}^{n-2} = \left(\frac{\sum_{i=1}^n |\lambda_i|}{n}\right)^{n-2} \leq \left(\frac{n|\lambda_1|}{n}\right)^{n-2}.$$

Therefore,

$$\left(\frac{n|a_0|c^{\frac{n-1}{2}}}{(n-1)c+s}\right)^{\frac{1}{n-2}} \leq |\lambda_1|.$$

We know that $|\det A| = |a_0|c^{\frac{n-1}{2}} = |\lambda_1||\lambda_2|\cdots|\lambda_n|$. Thus, $|a_0|c^{\frac{n-1}{2}} \leq |\lambda_1|^n$. Therefore, the following Theorem 2.8 holds.

Theorem 2.8. *Let A be an $n \times n$ matrix in $\mathcal{A}_c$ and let $\lambda_1, \ldots, \lambda_n$ be ordered eigenvalues of A so that $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{D}$. Then*

$$\max\left\{\left(|a_0|c^{\frac{n-1}{2}}\right)^{\frac{1}{n}},\ \left(\frac{n|a_0|c^{\frac{n-1}{2}}}{(n-1)c+s}\right)^{\frac{1}{n-2}}\right\} \leq |\lambda_1|.$$

Note that for any $n \times n$ nonsingular complex matrix B,

$$(|\lambda_1(B)|, \ldots, |\lambda_n(B)|) \prec_w (\sigma_1(B), \ldots, \sigma_n(B)).$$

That is, for an $n \times n$ matrix A in $\mathcal{A}_c$, $(|\lambda_1|, \ldots, |\lambda_n|) \prec_w (\sigma_1, \ldots, \sigma_n)$. So,

$$|\lambda_1| \leq \sigma_1, \ \sum_{i=1}^{n} |\lambda_i| \leq (n-2)\sqrt{c} + \sigma_1 + \sigma_n, \text{ and } |\lambda_n| \leq \frac{(n-2)\sqrt{c} + \sigma_1 + \sigma_n}{n}.$$

Lemma 2.9. *[4, 3.I.1.b] For* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R_+^n$, *the function*

$$\phi(\mathbf{x}) = (\sum_{i=1}^{n} x_i^r)^{\frac{1}{r}}$$

is Schur-convex if $r \geq 1$.

Theorem 2.10. *Let* A *be an* $n \times n$ *matrix in* $\mathcal{A}_c$ *and let* $\lambda_1, \ldots, \lambda_n$ *be ordered eigenvalues of* A *so that* $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{D}$. *Then*

$$|\lambda_n| \leq \sqrt{\frac{(n-1)c + s}{n}}.$$

Proof. By Lemma 2.9 and Lemma 2.6, we have

$$\bar{\lambda} \leq \left(\frac{|\lambda_1^2 + |\lambda_2|^2 + \cdots + |\lambda_n|^2}{n}\right)^{\frac{1}{2}} \leq \left(\frac{\sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2}{n}\right)^{\frac{1}{2}} = \sqrt{\frac{(n-1)c + s}{n}}.$$

Since $\bar{\lambda} = (\sum_{i=1}^{n} |\lambda_i|)/n$, the proof is completed.

Remark. Let $C = PU$ be the left polar decomposition of C with positive definite P and unitary U. In [1], P. van den Driessche and H. K. Wimmer yield the explicit formula for P and U in the left polar decomposition of a nonsingular companion matrix C where the coefficients of the polynomial $p(z)$ form the last row. The left polar decomposition of a W_c-companion matrix A is very closely that the companion matrix C.

Let A be an $n \times n$ matrix in $\mathcal{A}_c$. Set $\delta = \alpha\sqrt{c} + |a_0|\sqrt{c} + c$ and $\Gamma = \sqrt{c}\alpha I_{n-1} - \delta^{-1}(W_c\mathbf{d})(W_c\mathbf{d})^*$ where α is in (2.2). Then

$$P = \frac{1}{\alpha}\begin{bmatrix} \Gamma & W_c\mathbf{d} \\ (W_c\mathbf{d})^* & s + \sqrt{c}|a_0| \end{bmatrix} \tag{2.3}$$

is positive definite with $P = (AA^*)^{\frac{1}{2}}$ and $A = PU$ is the left polar decomposition of A for a unitary matrix

$$U = \frac{1}{\alpha}\begin{bmatrix} \frac{a_0}{|a_0|^2}(-\frac{\alpha}{\sqrt{c}} + \delta^{-1}||\mathbf{d}||^2 + 1)W_c\mathbf{d} & W_c(\frac{\alpha}{\sqrt{c}}I_{n-1} - \delta^{-1}\mathbf{d}\mathbf{d}^*) \\ \frac{a_0}{|a_0|}(\sqrt{c} + |a_0|) & \mathbf{d}^* \end{bmatrix} \quad (2.4)$$

The methodology of the proof follows very closely that of [1] for the companion matrix. Here is the proof as following as: for P defined in (2.3), one can easily verify that $P^2 = AA^*$. Therefore we have $P = (AA^*)^{\frac{1}{2}}$. Since A is nonsingular, we conclude that P is positive definite. To determine the unitary matrix U, we consider two cases:

Case (i) $\sigma_1^2 = c$ or $\sigma_n^2 = c$.

This is equivalent to $\mathbf{d} = \mathbf{0}$, by Lemma 2.1. Since $\mathbf{d} = \mathbf{0}$, we get $\Gamma = \sqrt{c}\alpha I_{n-1}$ and

$$P = (AA^*)^{\frac{1}{2}} = \mathrm{diag}\{\sqrt{c}, \cdots, \sqrt{c}, |a_0|\} = \frac{1}{\alpha}\begin{bmatrix} \Gamma & \mathbf{0} \\ \mathbf{0} & |a_0|^2 + \sqrt{c}|a_0| \end{bmatrix}.$$

Since $A = PU$ with P as above,

$$U = \begin{bmatrix} \mathbf{0} & \frac{1}{\sqrt{c}}W_c \\ \frac{a_0}{|a_0|} & \mathbf{0} \end{bmatrix}.$$

Then we can easily verify that $A = PU$.

Case (ii) $\sigma_1^2 < c < \sigma_n^2$.

This is equivalent to $\mathbf{d} \neq \mathbf{0}$. Let

$$\mathbf{v}_1 = \frac{1}{\sqrt{c}\ ||\mathbf{d}||}\begin{bmatrix} W_c\mathbf{d} \\ 0 \end{bmatrix},$$

and let $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be an $n \times 1$ matrix with $n-1$ zeros. Then,

$$AA^*\mathbf{v}_1 = c\mathbf{v}_1 + \sqrt{c}\ ||\mathbf{d}||\,\mathbf{v}_2 \quad \text{and} \quad AA^*\mathbf{v}_2 = \sqrt{c}\ ||\mathbf{d}||\,\mathbf{v}_1 + s\mathbf{v}_2.$$

So,

$$AA^*(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \mathbf{v}_2)\begin{bmatrix} c & \sqrt{c}\ ||\mathbf{d}|| \\ \sqrt{c}\ ||\mathbf{d}|| & s \end{bmatrix}.$$

For the computation of the square root of AA^*, it is sufficient to consider a symmetric 2×2 matrix. Let

$$H = \begin{bmatrix} c & \sqrt{c}||\mathbf{d}|| \\ \sqrt{c}||\mathbf{d}|| & s \end{bmatrix}.$$

Then $\det(zI - H) = f(z)$ and

$$H^{\frac{1}{2}} = \frac{\sqrt{c}}{\alpha} \begin{bmatrix} \sqrt{c} + |a_0| & ||\mathbf{d}|| \\ ||\mathbf{d}|| & \frac{\alpha^2}{\sqrt{c}} - |a_0| - \sqrt{c} \end{bmatrix}.$$

Now, we consider the eigenvalue c of AA^*. Let $\mathbf{y}_2, \cdots, \mathbf{y}_{n-1}$ be an orthonormal set of eigenvectors of AA^* satisfying $AA^*\mathbf{y}_i = c\mathbf{y}_i$ for $i = 2, \ldots, n-1$. Then for each $\mathbf{y}_i$ we have $\mathbf{y}_i^* = (\mathbf{x}_i^*, 0)$ and $(W_c\mathbf{d})^*\mathbf{x}_i = 0$. So, the matrix $V = (\mathbf{y}_2, \cdots, \mathbf{y}_{n-1}, \mathbf{v}_1, \mathbf{v}_2)$ is unitary, and

$$V^* AA^* V = \begin{bmatrix} cI_{n-2} & 0 \\ 0 & H \end{bmatrix}.$$

Since P is positive definite, each eigenvalue of P is positive and hence

$$P = (AA^*)^{\frac{1}{2}} = \left(V \begin{bmatrix} cI_{n-2} & 0 \\ 0 & H \end{bmatrix} V^* \right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} P &= \sqrt{c}I_n + (\mathbf{v}_1, \mathbf{v}_2)(H^{\frac{1}{2}} - \sqrt{c}\, I_2) \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \end{bmatrix} \\ &= \frac{1}{\alpha} \begin{bmatrix} \Gamma & W_c\mathbf{d} \\ (W_c\mathbf{d})^* & s + \sqrt{c}\, |a_0| \end{bmatrix}. \end{aligned}$$

Since A is nonsingular,

$$A^{-1} = \begin{bmatrix} -\frac{(W_c\mathbf{d})^*}{c} a_0 & \frac{1}{a_0} \\ \frac{1}{c} W_c^* & 0 \end{bmatrix}.$$

For the unitary factor of $A = PU$ we have $U = P(A^{-1})^*$. Hence

$$U = \frac{1}{\alpha} \begin{bmatrix} \frac{a_0}{|a_0|^2}(-\frac{\alpha}{\sqrt{c}} + \delta^{-1}||\mathbf{d}||^2 + 1)W_c\mathbf{d} & W_c(\frac{\alpha}{\sqrt{c}} I_{n-1} - \delta^{-1}\mathbf{d}\mathbf{d}^*) \\ \frac{a_0}{|a_0|}(\sqrt{c} + |a_0|) & \mathbf{d}^* \end{bmatrix}.$$

The proof is completed.
This conclude our argument. For example,

$$A=\begin{bmatrix}0&1&1&1&1\\0&1&-1&1&-1\\0&1&1&-1&-1\\0&1&-1&-1&1\\1&2&2&2&2\end{bmatrix},$$

then

$$P=\frac{1}{5}\begin{bmatrix}6&0&0&0&8\\0&10&0&0&0\\0&0&10&0&0\\0&0&0&10&0\\8&0&0&0&19\end{bmatrix},\qquad U=\frac{1}{5}\begin{bmatrix}-4&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}\\0&\frac{5}{2}&-\frac{5}{2}&\frac{5}{2}&-\frac{5}{2}\\0&\frac{5}{2}&\frac{5}{2}&-\frac{5}{2}&-\frac{5}{2}\\0&\frac{5}{2}&-\frac{5}{2}&-\frac{5}{2}&\frac{5}{2}\\3&2&2&2&2\end{bmatrix}.$$

3. Other Results

Norms can be thought of a generalization of Euclidean length, but the study of norms has a lot more impact, because it is necessary for a proper formulation of notions such as power series of matrices and is essential in the analysis and assessment of algorithms for numerical computations. Even though any computation of a matrix norm is complicated, we are going to close our study of companion matrix with a matrix norm for an $n \times n$ matrix G defined by

$$G=\begin{bmatrix}\mathbf{x} & I_{n-1}\\ a_0 & \mathbf{d}^*\end{bmatrix}, \tag{3.1}$$

where $a_0 \neq 0$, $\mathbf{d}^* = (a_1, \ldots, a_{n-1})$, $\mathbf{x} = (b, 0, \ldots, 0)^T$. Let $\mathcal{G}$ be the set of all matrix as in (3.1) for a complex number b.

Theorem 3.11. *Let G be an $n \times n$ matrix in $\mathcal{G}$. Then*

$$|||G|||_2 = \max\{\sqrt{z} \mid z = 1,\ z^3 - ez^2 + kz + l = 0\},$$

where $e = s + |b|^2 + 2$, $k = (|b|^2+1)(s+1) - |\bar{a}_0 b + a_1|^2 + |a_1|^2 + |a_0|^2$, and $l = -|\bar{b}a_1 - \bar{a}_0|^2$.

Proof. Since

$$GG^* = \begin{bmatrix}\operatorname{diag}(|b|^2+1, 1, \ldots, 1) & \bar{a}_0\mathbf{x} + \mathbf{d}\\ (\bar{a}_0\mathbf{x}+\mathbf{d})^* & s\end{bmatrix},$$

by Schur complements,

$$\begin{aligned}\det(zI_n - GG^*) &= (z-1)^{n-2}(z-(|b|^2+1))[(z-s)-(-(\bar{a}_0\mathbf{x}+\mathbf{d})^* \\ &\quad (\mathrm{diag}\{z-(|b|^2+1), z-1, \ldots, z-1\})^{-1}(-(\bar{a}_0\mathbf{x}+\mathbf{d}))] \\ &= (z-1)^{n-2}(z-(|b|^2+1))(z-s)-(\bar{a}_0\mathbf{x}+\mathbf{d})^* \\ &\quad \mathrm{diag}\{\tfrac{1}{z-(|b|^2+1)}, \tfrac{1}{z-1}, \ldots, \tfrac{1}{z-1}\}(\bar{a}_0\mathbf{x}+\mathbf{d})\end{aligned}$$

Thus, we have

$$\det(zI_n - GG^*) = (z-1)^{n-3}(z^3 - ez^2 + kz + l),$$

where $e = s + |b|^2 + 2$, $k = (|b|^2+1)(s+1) - |\bar{a}_0 b + a_1|^2 + |a_1|^2 + |a_0|^2$, and $l = -|\bar{b}a_1 - \bar{a}_0|^2$.

The proof is completed.

If $b = 0$, then the matrix G is in $\mathcal{A}_1$, that is, G is a companion matrix C. And, from Corollary 2.3, we know that

$$|||C|||_2 = \frac{\sqrt{s+1+2|a_0|} + \sqrt{s+1-2|a_0|}}{2}.$$

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