Some Properties of Seperation Axioms on Nearness Structures

Bang Eun-sook

Nearness 構造의 分離公理에 관한 몇가지 性質에 관하여

方 銀 淑

Summary

We construct the completion of an N-space by means of γ-coclusters, and our results show that

1. Concrete N1-structures are a proper tool to investigate strict extensions,
2. Contigual N1-structures are a proper tool to investigate strict compactifications,
3. N3-structures are a proper tool to investigate regular extensions.

INTRODUCTION

The concept of neaness was introduced [1] as a unification of various concepts of topological structures. In fact, it has been shown that the categories of topological R_0-spaces, uniform spaces, proximity spaces and contiguity spaces are nicely embedded in the category of nearness spaces [2].

A concept of completness is available for nearness spaces which generalizes the concept of completness in uniform spaces. Moreover every nearness space has a completion. In [1], this completion of a nearness space (X,ξ) has been constructed by means of ξ-clusters.

Now, we try to construct the completion of a nearness space by means of γ-coclusters and apply them, among others, to the study of extensions of spaces.

In the present paper, the most results are analogous to the important paper "A Concept of Nearness" by H. Herrlich.

1. NOTATION, TERMINOLOGY AND BASIC CONCEPTS.

X is a set, $P^1X = PX$ denotes the power set of X and $P^{n+1}X$ denotes the power set of P^nX.

Small Latin letters x,y,z, ... usually denote elements of X.

Latin captitals A,B,C, ... usually denote subsets of X.

Bold-faced Latin capitals **A**,**B**,**C**, ... usually denote subsets of PX.

Small Greek letters ξ,η,ω, ... usually denote subsets of P^2X.

Capital Greek letters Ω,Λ, ... usually denote subsets of P^3X.

For any subset ξ of P^2X, the following abbreviations are used: "**A** is near" or ξ**A** for **A**$\in\xi$, "**A** is far" or $\bar{\xi}$**A** for **A**$\in(P^2X-\xi)$, $A\xi$**A** for $\xi(\{A\}\cup$**A**$)$, $A\xi B$ for $\xi\{A,B\}$, $Cl_\xi A$ for $\{x\in X: \{x\}\xi A\}$

1.1. DEFINITIONS.

sec **A** $=\{B\subset X: \forall A\in$**A** $A\cap B\neq\phi\}$.

stack $\mathbf{A} = \{B \subset X : \exists A \in \mathbf{A}\ A \subset B\}$.

$\mathbf{A} \vee \mathbf{B} = \{A \cup B : A \in \mathbf{A} \text{ and } B \in \mathbf{B}\}$.

$\mathbf{A} \wedge \mathbf{B} = \{A \cap B : A \in \mathbf{A} \text{ and } B \in \mathbf{B}\}$.

$\mathbf{A} \ll \mathbf{B} \Longleftrightarrow \forall A \in \mathbf{A}\ \exists B \in \mathbf{B}\ A \subset B \Longleftrightarrow$ $\mathbf{A}$ refines $\mathbf{B}$.

$\mathbf{A} < \mathbf{B} \Longleftrightarrow \forall A \in \mathbf{A}\ \exists B \in \mathbf{B}\ B \subset A \Longleftrightarrow$ $\mathbf{A}$ corefines $\mathbf{B}$.

$\mathbf{A} \sim \mathbf{B} \Longleftrightarrow (\mathbf{A} < \mathbf{B}$ and $\mathbf{B} < \mathbf{A})$.

$\mathbf{A}$ is called a stack in X iff $\mathbf{A}$ = stack $\mathbf{A}$.

$\mathbf{A}$ is called a grill in X iff $\phi \neq \mathbf{A} \neq PX$ and $A \cup B \in \mathbf{A} \Longleftrightarrow (A \in \mathbf{A}$ or $B \in \mathbf{B})$.

$\mathbf{A}$ is called a filter in X iff $\phi \neq \mathbf{A} \neq PX$ and $A \cap B \in \mathbf{A} \Longleftrightarrow (A \in \mathbf{A}$ and $B \in \mathbf{A})$.

1.2. DEFINITIONS. If x is a point and $\mathbf{A}$ is a collection of subsets to a topological space (X,cl) then

(a) x is an adherencepoint of $\mathbf{A}$ iff $x \in \bigcap\{clA : A \in \mathbf{A}\}$.

(b) $\mathbf{A}$ converges to x iff the neighbourhoodfilter of x corefines $\mathbf{A}$.

1.3. COROLLARY.

(1) x is an adherencepoint of $\mathbf{A}$ iff sec $\mathbf{A}$ converges to x.

(2) $\mathbf{A}$ converge to x iff x is an adherencepoint of sec $\mathbf{A}$.

1.4. PROPOSITIONS (characterizations of sec, stack and $\sim$).

(1) sec $\mathbf{A} = \{B \subset X : X - B \notin$ stack $\mathbf{A}\}$.

(2) stack $\mathbf{A} = \sec^2 \mathbf{A}$.

(3) $\mathbf{A} \sim \mathbf{B} \Longleftrightarrow$ sec $\mathbf{A}$ = sec $\mathbf{B} \Longleftrightarrow$ stack $\mathbf{A}$ = stack $\mathbf{B}$.

(4) $\sec^3 \mathbf{A} = \sec \mathbf{A}$ (i.e. sec $\mathbf{A}$ is a stack).

(5) stack is a topological closure operator on PX.

1.5. PROPOSITIONS. Let SX be the set of all stacks in X, and let $\mathbf{A}$ and $\mathbf{B}$ be elements of SX. Then

(1) $\mathbf{A} < \mathbf{B} \Longleftrightarrow \mathbf{A} \subset \mathbf{B} \Longleftrightarrow \sec \mathbf{B} \subset \sec \mathbf{A}$.

(2) $\mathbf{A} \sim \mathbf{B} \Longleftrightarrow \mathbf{A} = \mathbf{B}$.

(3) $\mathbf{A} \vee \mathbf{B} = \mathbf{A} \cap \mathbf{B}$.

(4) $\mathbf{A} = \sec \mathbf{B} \Longleftrightarrow \mathbf{B} = \sec \mathbf{A}$.

(5) $\mathbf{A}$ is a filter $\Longleftrightarrow$ sec $\mathbf{A}$ is a grill.

1.6. DEFINITION. A pair (X,ξ) is called a nearness space or N-space iff the following conditions satisfied:

(N1) If $\mathbf{A} < \mathbf{B}$ and $\xi\mathbf{B}$ then $\xi\mathbf{A}$

(N2) If $\cap \mathbf{A} \neq \phi$, then $\xi\mathbf{A}$

(N3) $\phi \neq \xi \neq P^2X$

(N4) If $\xi(\mathbf{A} \vee \mathbf{B})$, then $\xi\mathbf{A}$ or $\xi\mathbf{B}$

(N5) if $\xi\{Cl_\xi A : A \in \mathbf{A}\}$ then $\xi\mathbf{A}$

1.7. DEFINITION. If (X,ξ) and (Y,η) are N-spaces, then $f:(X,\xi) \to (Y,\eta)$ is called a nearness preserving map or an N-map, if $\xi\mathbf{A}$ implies $\eta(f\mathbf{A})$-where $f\mathbf{A} = \{fA : A \in \mathbf{A}\}$.

1.8. DEFINITION. Let ξ be a nearness structure on X. Then

(1) $\bar{\xi} = P^2X - \xi$ is called the farness structure induced on X by ξ;

(2) $\mu = \mu_\xi = \{\mathbf{A} \subset PX : \bar{\xi}\{X - A : A \in \mathbf{A}\}\}$ is called the covering structure induced on X by ξ;

(3) $\gamma = \gamma_\xi = \{\mathbf{A} \subset PX : \forall \mathbf{B} \in \mu,\ \mathbf{B} \cap \text{stack } \mathbf{A} \neq \phi\}$ is called the merotopic structure induced on X by ξ.

The above three structure on an N-space(X,ξ) which are associated with ξ. Obviously, ξ can be recovered from each of the structure $\bar{\xi}$, μ and γ.

1.9. PROPOSITIONS. Let ξ be a nearness structure on X and let $\bar{\xi}$, η and γ be the associated structures. Then

(1) $\mathbf{A} \in \xi$ iff sec $\mathbf{A} \in \gamma$.

(2) $\mathbf{A} \in \gamma$ iff sec $\mathbf{A} \in \xi$.

(3) $\mathbf{A} \in \mu$ iff $\forall \mathbf{B} \in \xi, \mathbf{A} \cap \sec \mathbf{B} \neq \phi$.

(4) $\mathbf{A} \in \xi$ iff $\forall \mathbf{B} \in \mu, \mathbf{B} \cap \sec \mathbf{A} \neq \phi$.

(5) $\mathbf{A} \in \gamma$ iff $\forall \mathbf{B} \in \bar{\xi}\ \exists A \in \mathbf{A}\ \exists B \in \mathbf{B}\ A \cap B = \phi$.

(6) Equivalents are :

(a) $x \in Cl_\xi A$. (b) $\sec\{A, \{x\}\} \in \gamma$

(c) $\{X - A, X - \{x\}\} \notin \mu$

(7) If $f:(X,\xi) \to (Y,\eta)$ is a map between N-spaces then the following conditions are equivalent:

(a) $\xi\mathbf{A} \to \eta(f\mathbf{A})$ (b) $\bar{\eta}\mathbf{B} \to \bar{\xi}(f^{-1}\mathbf{B})$

(c) $\gamma_\xi\mathbf{A} \to \gamma_\eta(f\mathbf{A})$ (d) $\mu_\eta\mathbf{B} \to \mu_\xi(f^{-1}\mathbf{B})$

1.10. PROPOSITIONS. Let γ be a subset of P^2X:

(S1) if $\mathbf{A} < \mathbf{B}$ and $\gamma\mathbf{A}$ then $\gamma\mathbf{B}$.

(S2) $\forall x \in X, \gamma\{\{x\}\}$.

(S3) $\phi \neq \gamma \neq P^2X$.

(S4) if $\gamma(\mathbf{A} \cup \mathbf{B})$ then $\gamma\mathbf{A}$ or $\gamma\mathbf{B}$.

(S5) $\gamma(\sec\{ClA : A \in \mathbf{A}\}) \to \gamma(\sec \mathbf{A})$-where $ClA = \{x \in X : \gamma(\sec\{A, \{x\}\})\}$.

1.11. REMARK. A merotopic structure γ on X induces a topology on X. Indeed, the interior operator is defined by Int $A = \{x \in X : \sec\{\{x\}, X - A\} \notin \gamma\}$. A subset of a nearness space X shall be refered to as an open set if it is open in the induced topology. On the other hand, if we define $ClA = \{x \in X : \sec\{\{x\}, A\} \in \gamma\}$ for a subset A of X, then obviously $IntA = X - Cl(X - A)$ and therfore Cl is the Kuratowski's closure operator on the induced topological space.

1.12. DEFINITIONS. An N-space (X,ξ) is called a topological N-space iff the following equivalent conditions are satisfied:

(T) If $\xi\mathbf{A}$ then $\cap\{Cl_\xi A: A\in\mathbf{A}\}\neq\phi$.

(T') If $\gamma\mathbf{A}$ then $\mathbf{A}$ converges.

An N-space (X,ξ) is called a contigual N-space iff the following equivalent conditions are satisfied:

(C) If every finite subset of $\mathbf{A}$ belongs to ξ then $\mathbf{A}$ belongs to ξ.

(C') If $\bar{\xi}\mathbf{A}$ then there exists finite subset $\mathbf{B}$ of $\mathbf{A}$ with $\bar{\xi}\mathbf{B}$.

1.13. NOTATIONS. The category of N-spaces and N-maps is denoted by *Near*. The fully subcategory of *Near* whose objects are topological N-spaces (contigual N-spaces, resp.) is denoted by *T-Near* (*C-Near*, resp.).

1.14. THEOREM

(1) *T-Near* is a bicoreflective subcategory of *Near*. Let $(X,\xi)\in$ *Near* and $\xi_t=\{\mathbf{A}\subset PX:\cap\{Cl_\xi A:A\in\mathbf{A}\}\neq\phi\}$.
Then the map $id_X:(X,\xi_t)\to(X,\xi)$ is the *T-Near* coreflection of (X,ξ).

(2) C-Near is a bireflective subcategory of *Near*.
Let $(X,\xi)\in$ *Near* and $\xi_c=\{\mathbf{A}\subset PX:\forall\mathbf{B}\subset\mathbf{A}\ (\mathbf{B}\text{ finite}\to\xi\mathbf{B})\}$.
Then the map $id_X:(X,\xi)\to(X,\xi_c)$ is the *C-Near* reflection of (X,ξ).

1.15. THEOREM. A nearness space is topological and contigual iff it is compact topological space.

1.16. DEFINITION. An N-space is called compact iff it is topological and contigual.

1.17. DEFINITION. An N-space is called an N1-space iff the following equivalent conditions are satisfied:

(1) If $\{x\}\xi\{y\}$ then $x=y$.

(2) If $\{\{x,y\}\}\in\gamma$ then $x=y$.

1.18. DEFINITION. If (X,ξ) is an N-space, $\mathbf{A}\subset PX$, $A\subset X$ and $B\subset X$ then

(1) $A<_\xi B$ iff $A\bar{\xi}(X\text{-}B)$.

(2) $\mathbf{A}(<_\xi)=\{B\subset X:\exists A\in\mathbf{A}\ A<_\xi B\}$.

1.19. COROLLARY. If (X,ξ) is an N-space and $\mathbf{A}\subset PX$ then $\sec(\mathbf{A}(<_\xi))=\{B\subset X:\forall A\in\mathbf{A},\ A\xi B\}$.

1.20. DEFINITION. An N-space (X,ξ) is called regular iff the following equivalent conditions are satisfied:

(1) If $\xi\mathbf{A}(<_\xi)$ then $\xi\mathbf{A}$.

(2) If $\gamma\mathbf{A}$ then $\gamma\mathbf{A}(<_\xi)$.

(3) $\gamma\mathbf{A}$ iff $\xi\{B\subset X:\forall A\in\mathbf{A}, A\xi B\}$.

(4) $\xi\mathbf{A}$ iff $\gamma\ B\subset B:\forall A\in\mathbf{A},\ A\xi B\}$.

1.21. PROPOSITION. If (X,ξ) is a regular N-space and $\mathbf{A}\in\xi\cap\gamma$ then

(1) $\sec(\mathbf{A}(<_\xi))=\xi(\mathbf{A})$,

(2) $\xi(\mathbf{A})=\{B\subset X:\forall A\in\mathbf{A},\ A\xi B\}$ is the unique ξ-cluster containing $\mathbf{A}$,

(3) If $\mathbf{A}$ is a γ-filter then $\mathbf{A}(<_\xi)$ is the unique minimal γ-filter contained in $\mathbf{A}$.

2. COMPLETENESS

From now on, our nearness spaces always means N1 spaces.

2.1. DEFINITIONS. Let (X,ξ) be an N-space. A nonempty subset $\mathbf{A}$ of PX is called:

(1) A ξ-cluster iff $\mathbf{A}$ is a maximal element of the set ξ, ordered by inclusion.

(2) A γ-cocluster iff $\mathbf{A}$ is a minimal element of the set $\{\mathbf{B}\in\gamma:\mathbf{B}=\text{stack }\mathbf{B}\}$, ordered by inclusion.

(3) A γ-filter -or Cauchy filter- if $\mathbf{A}$ is a filter and $\mathbf{A}\in\gamma$.

2.2. PROPOSITION. Let (X,ξ) be an N-space. For nonempty stacks $\mathbf{A}$ in X then

(1) $\mathbf{A}$ is a ξ-cluster iff $\sec\mathbf{A}$ is a γ-cocluster,

(2) $\mathbf{B}=\sec\mathbf{A}$ is a γ-cocluster implies that $\mathbf{B}$ is a minimal γ-filter,

(3) $\mathbf{B}$ is a minimal γ-filter implies that $\mathbf{B}$ is a γ-filter.

2.3. REMARK

(1) If (X,ξ) is an N-space and $x\in X$ then

(a) $\xi(x)=\{A\subset X:x\in Cl_\xi A\}$ is a ξ-cluster,

(b) the neighbourhood filter $U(x)$ of x is a γ-cocluster.

(2) If (X,ξ) is contigual or topological, then γ-coclusters are precisely minimal γ-filters. (i.e. minimal Cauchy filters)

(3) If (X,ξ) is topological then the ξ-clusters are precisely the collections $\xi(x)$ and the γ-coclusters are precisely the neighbourhood filters $U(x)$.

In the above remark, the concept of γ-coclusters seems to be more intuitive than that of ξ-clusters.

2.4. DEFINITION. An N-space (X,ξ) is said to be complete iff the following equivalent conditions:

(1) If $\mathbf{A}$ is any ξ-cluster then $\mathbf{A}$ has an adherencepoint. (i.e. $\mathbf{A} = \xi(x)$ for some $x \in X$)

(2) Every γ-cocluster converges.

2.5. LEMMA. If (X,ξ) is contigual then

(1) for each $\xi\mathbf{A}$ there exists a ξ-cluster $\mathbf{B}$ with $\mathbf{A}\subset\mathbf{B}$,

(2) for each $\gamma\mathbf{B}$ there exists a γ-cocluster $\mathbf{A}$ with $\mathbf{A} < \mathbf{B}$,

(3) for each γ-filter $\mathbf{A}$ there exists a γ-cocluster $\mathbf{B}$ with $\mathbf{B}\subset\mathbf{A}$.

PROOF. (1) Let $\eta=\{\xi\mathbf{B}:\mathbf{A}\subset\mathbf{B}\}$ and define the relation "$\leqslant$" on η by $\mathbf{B}\leqslant\mathbf{B}'$ iff $\mathbf{B}\subset\mathbf{B}$, for any $\mathbf{B}$, $\mathbf{B}'$ in η. Then $(\eta,\leqslant)$ is a poset and furthermore, it is inductive.

By Zorn's Lemma, η has a maximal element, say $\mathbf{D}$. Thus $\eta\mathbf{D}$; $\mathbf{A}\subset\mathbf{D}\in\xi$, so that $\mathbf{D}$ is a ξ-cluster.

(2), (3) It is obvious by 2.2.

2.6. PROPOSITION.

(1) Every topological N-space is complete.

(2) A contigual N-space is complete iff it is compact.

PROOF. (1) It follows from 2.3(3)

(2) The sufficiency is immediate from (1) and 1.16. For the necessity, let $\xi\mathbf{A}$, By 2.5(1), there exists a ξ-cluster $\mathbf{B}$ with $\mathbf{A}\subset\mathbf{B}$.

Then $\mathbf{B}$ has an adherence point, and so is $\mathbf{A}$. Thus $\cap\{Cl_\xi A:A\in\mathbf{A}\}\neq\phi$.

Hence (X,ξ) is a topological N-space and it is compact.

2.7. PROPOSITION. Let (X,ξ) be an N-space and $\mathbf{F}$ be a γ-cocluster. Then $F\in\mathbf{F}$ iff there exists $\mathbf{B}\in\mu$ with $\cup(\mathbf{F}\cap\mathbf{B})\subset F$, where μ is introduced in 1.8(2).

PROOF. See [10].

2.8. COROLLARY. Let $\mathbf{F}$ be a γ-cocluster. Then $\mathbf{F}$ is the filter generated by $\{\cup(\mathbf{F}\cap\mathbf{B}):\mathbf{B}\in\mu\}$.

2.9. COROLLARY. Let $\mathbf{F}$ be a γ-cocluster. Then $F\in\mathbf{F}$ iff $IntF\in\mathbf{F}$ and therefore $\mathbf{F}$ is an open filter.

3. COMPLETION

3.1. DEFINITIONS. Let (X,ξ) be an N-space. Denote by

(1) $Y = \{\mathbf{A}:\mathbf{A}$ is a γ-cocluster which does not converge in $X\}$.

(2) $X^* = X\cup Y$ (i.e. X^* is the disjoint union of X and Y)

(3) $\hat{B} = \{\mathbf{A}\in Y:B\in\mathbf{A}\}\cup Int_X B$, for each subset B of X.

We shall show that X^* admits a suitable merotopic structure and $\hat{B}$ is the largest open set in X^* whose intersection with X is $Int_X B$.

3.2. THEOREM. Let (X,ξ) be an N-space. For $\Omega\subset PX^*$, let $\Omega\in\gamma^*$ iff $\{A\subset X:\hat{A}\in stack\Omega\}\in\gamma$. Then γ^* is a merotopic structure on X^*.

PROOF. (S1)-(S4) are straightforward and are omitted. To prove (S5), we need following Lemma.

3.3. LEMMA. Under the same notation as that in 3.2, $x\in Cl_{X^*}\omega$ iff $\hat{A}\cap\omega\neq\phi$ for each $A\subset X$ with $x\in\hat{A}$, $\omega\subset X^*$. That is $\{\hat{A}:A\subset X\}$ is a basis for open sets in (X^*,γ^*).

PROOF. For necessity, let $x\in Cl_{X^*}\omega$. By definition, $sec_{X^*}\{\{x\},\omega\}\in\gamma^*$, hence $\mathbf{D} = \{A\subset X:\hat{A}\in sec_{X^*}\{\{x\},\omega\}\}\in\gamma$. Observe that $\mathbf{D}$ is a γ-cocluster and so is the result.

Conversely, assume that $\hat{A}\cap\omega\neq\phi$ for every $A\subset X$ with $x\in\hat{A}$. Then $\{A\subset X:x\in\hat{A}\}\subset\{A\subset X:\hat{A}\in sec\{\{x\},\omega\}\}$ and hence $\{A\subset X:\hat{A}\in sec\{\{x\},\omega\}\}\in\gamma$ or $sec\{\{x\},\omega\}\in\gamma^*$, Thus $x\in Cl_{X^*}\omega$.

PROOF. (of (S5) in Theorem 3.2.). Suppose that $sec\{Cl_{X^*}\omega:\omega\in\Omega\}\in\gamma^*$ Define $\mathbf{B} = \{A\subset X:\hat{A}\in sec\{Cl_{X^*}\omega:\omega\in\Omega\}\}$ and $\mathbf{D} = \{A\subset X:\hat{A}\in sec\Omega\}$. Then $\mathbf{B}\subset\mathbf{D}$, for if $A\in\mathbf{B}$ then $\hat{A}\cap Cl_{X^*}\omega\neq\phi$ for all $\omega\in\Omega$. Let $x\in\hat{A}\cap Cl_{X^*}\omega$. Then by 3.2, $\hat{A}\cap\omega\neq\phi$ for all $\omega\in\Omega$. This fact together with $\gamma\mathbf{B}$ implies $\gamma\mathbf{D}$ or $\gamma^*(sec\Omega)$.

3.4. PROPOSITION. If Ω is a γ^*-cocluster, then $\{A\subset X:\hat{A}\in stack\Omega\}$ is a γ-cocluster.

PROOF. Let $\mathbf{A} = \{A\subset X:\hat{A}\in stack\Omega\}$. Then $\mathbf{A} = \{A\subset X:\hat{A}\in\Omega\}\in\gamma$ and obviousley $\mathbf{A}$ is a stack. If $A\in\mathbf{A}$, then there exists a $\omega\in\Omega$ with $\omega\subset\hat{A}$. Since Ω is a γ^*-cocluster, there exists $\Lambda\in\mu^*$ with $\cup(\Omega\cap\Lambda)\subset\omega$, where μ^* is the associated covering structure with γ^*. Thus we have $\cup(\Omega\cap\Lambda)\cap X\subset\omega\cap X\subset\hat{A}\cap X = Int_X A\subset A$. But $\Lambda\in\mu^*$ iff $\mathbf{B} = \{A\subset X: \exists\lambda\in$ with $\hat{A}\subset\lambda\}$. Now, to prove $\mathbf{A}$ is a γ-cocluster, (by 2.7) we show that $\cup(\mathbf{A}\cap\mathbf{B})\subset A$. Let $D\in\mathbf{A}\cap\mathbf{B}$. Then $\hat{D}\in\Omega$ and $\hat{D}\subset\lambda$ for some $\lambda\in\Lambda$.

Thus $\lambda\in\Omega$ and $\lambda\in\Omega\cap\Lambda$ which imply $D\subset A$. This completes the proof.

3.5. THEOREM. (X^*,γ^*) is a complete N1-space.

PROOF. Let Ω be a γ^*-cocluster. Then $\mathbf{A} = \{A \subset X: \hat{A} \in \Omega\}$ is a γ-cocluster.

Case 1). $\mathbf{A} \in Y$, i.e. $\mathbf{A}$ does not converge in X. Then $\{\hat{A}: A \in \hat{\mathbf{A}}\} = \{\hat{A}: A \in \mathbf{A}\}$ is the open neighborhood filter of $\mathbf{A}$ and corefines Ω, which implies that Ω converges to $\mathbf{A}$.

Case 2). Suppose $\mathbf{A}$ converges to x for some $x \in X$. Then $\mathbf{A} = \{A \subset X: x \in Int_X A$ and obviousely $\{\hat{A}: x \in \hat{A}\} = \{\hat{A}: A \in \mathbf{A}\}$ corefines Ω, hence Ω converges to x. Hence (X^*, γ^*) is complete. To prove (X^*, γ^*) is N1, let $\{\{\mathbf{A}, \mathbf{B}\}\} \in \gamma^*$, then $\gamma(\mathbf{A} \cap \mathbf{B})$. Since γ-coclusters are minimal elements γ-$\{\phi\}$, this implies $\mathbf{A}=\mathbf{B}$. This completes the proof.

3.6. THEOREM. (X,γ) is dense in (X^*, γ^*).

PROOF. By 3.3, it sufficies to show that $\hat{B} \cap X = Int_X B \neq \phi$ for any $B \subset X$ with $\hat{B} \neq \phi$.

Assume that $\hat{B} \neq \phi$ but $Int_X B = \phi$. Then for any γ-cocluster $\mathbf{A}$, $Int_X B \notin \mathbf{A}$ and so $B \notin \mathbf{A}$. Therefore $\{\mathbf{A} \in Y: B \in \mathbf{A}\} \cup Int_X B = \phi$ which contradicts $\hat{B} \neq \phi$.

3.7. THEOREM. Let $j:(X,\gamma) \to (X^*, \gamma^*)$ be an inclusion map. Then j is an embedding.

PROOF. Let $\mathbf{A} \subset PX$, We shall show that $\gamma\mathbf{A}$ iff $\gamma^*\mathbf{A}$. For sufficiency let $\gamma^*\mathbf{A}$. Then $\mathbf{B} = \{B \subset X: \hat{B} \in stack\,\mathbf{A}\} \in \gamma$ and $\mathbf{B} < \mathbf{A}$. Hence $\gamma\mathbf{A}$.

Conversely, suppose $\gamma\mathbf{A}$ and let $\mathbf{B} = \{B \subset X: \hat{B} \in stack\,\mathbf{A}\}$.

Assume that $\mathbf{B} \notin \gamma$. By 1.8 (3), $stack\,\mathbf{B} \cap \mathbf{D} = \mathbf{B} \cap \mathbf{D} = \phi$ for some $\mathbf{D} \in \mu$.

Since $Int\mathbf{D} \in \mu$ and $Int\mathbf{D} \cap stack\,\mathbf{A} = \phi$, hence $\mathbf{A} \notin \gamma$ which is a contradiction.

3.8. THEOREM. (X^*, γ^*) is the completion of (X,γ).

PROOF. It is immediate from 3.2, 3.5, 3.6 and 3.7.

In [2], a completion (X^*, ξ^*) of an N-space(X,ξ) has been constructed by the following: Let

(1) X^* be the set of all ξ-clusters,

(2) $\xi^* = \{\Omega \subset PX^*: \cup\{\cap\omega: \omega \in \Omega\} \in \xi\}$,

(3) $j: X \to X^*$ the map defined by $j(x) = \xi(\{x\})$.

Then $j:(X,\xi) \to (X^*, \xi^*)$ is the completion of (X,ξ).

3.9. REMARK. If (X,ξ) is a topological N-space, then $(X^*, \xi^*) = (X,\xi)$.

3.10. THEOREM. (See [1] & [2]) An N-space (X,ξ) is

(1) regular iff (X^*, ξ^*) is regular

(2) contigual iff (X^*, ξ^*) is contigual iff (X^*, ξ^*) is compact.

4. MAIN RESULTS

4.1. DEFINITION. A N-map $f:(X,\xi) \to (Y,\eta)$ is called

(1) an N-embedding iff $f: X \to Y$ is injective and $\xi\mathbf{A} \Leftrightarrow \eta(f\mathbf{A})$,

(2) dense iff $Cl_\eta(fX) = Y$,

(3) a topological extension iff it is a dense topological embedding and (X,ξ) and (Y,η) are topological,

(4) a T1 - extension iff it is a topological extension and (X,ξ) and (Y,η) are T1-spaces.

(5) a strict extension iff it is T1-extension and $\{Cl_\eta fA: A \subset X\}$ is a base for the closed sets in (Y,η)

(6) a compactification iff it is a T1-extension and (Y,η) is compact.

4.2. REMARK(1). The completion $j:(X,\xi) \to (X^*, \xi^*)$ of a N1-space (X,ξ) is a dense N-embedding.

(2) Any dense topological embedding of (X,ξ) into a regular T1-space (Y,η) is a strict extension of (X,ξ).

4.3. DEFINITION. Extensions $f:(X,\xi) \to (Y,\eta)$ and $f':(X,\xi) \to (Y',\eta')$ of (X,ξ) are called equivalent iff there exists a homeomorphism $h:(Y,\eta) \to (Y',\eta')$ with $f' = h \circ f$

In the following, all N-spaces are again supposed to be N1-spaces.

4.4. DEFINITION. An N-space (X,ξ) is called concrete iff for each $\xi\mathbf{A}$ there exists a ξ-cluster $\mathbf{B}$ with $\mathbf{A} \subset \mathbf{B}$.

4.5. THEOREM. If (X,ξ) is an N-space, then (X,ξ) is concrete iff (X^*, ξ^*) is topological.

PROOF. See [3].

4.6. COROLLARY. Let (X,ξ) be an N-space. If (X,ξ) is contigual or topological or regular, then it is concrete.

PROOF. It's immediate from 2.5. and 4.5..

4.7. PROPOSITION. If $j:(X,\xi) \to (X^*, \xi^*)$ is the completion of (X,ξ), then $j:(X,\xi_t) \to (X^*, (\xi^*)_t)$ is a strict extension of (X,ξ_t).

PROOF. It follows that the induced topological space of (X^*, γ^*) is the strict extension of the induced topological space of (X,γ) with all γ-cocluster as filter trace.

4.8. THEOREM. If (X,ξ) is a concrete N-space then $j:(X,\xi_t) \to (X^*,\xi^*)$ is a strict extension of (X,ξ_t). Vice versa, for any strict extension $f:(X,\zeta) \to (Y,\eta)$ of a topological N-space (X,ζ) there exists precisely one concrete N-structure ξ on X, namely

$$\xi = \{ \mathcal{A} \subset PX : \cap \{Cl_\eta fA : A \in \mathcal{A}\} \neq \phi \}$$

such that $j:(X,\xi_t) \to (X^*, \xi^*)$ and $f:(X,\zeta) \to (Y,\eta)$ are equivalent extensions of $(X,\xi_t)=(X,\zeta)$. In particular,

(1) (X,ξ) is contigual iff (Y,η) is a compact space.

(2) (X,ξ) is regular iff (Y,η) is a regular space.

PROOF: (a) By 4.5, obviously $j:(X,\xi_t) \to (X^*,\xi^*)$ is a strict extension $f(X,\xi_t)$.

(b) Conversely, let $f:(X,\zeta) \to (Y,\eta)$ be a strict extension. If $\xi = \{\mathcal{A} \subset PX : \cap \{Cl_\eta fA : A \in \mathcal{A}\} \neq \phi\}$, then (X,ξ) is a N1-space and $\xi_t = \zeta$. Let $j:(X,\xi) \to (X^*,\xi^*)$ be the completion of (X,ξ) and let $j:(X,\xi_t) \to (X^*,(\xi^*)_t)$ be the corresponding extension. For each $y \in Y$ define $h(y) = \{A \subset X : y \in Cl_\eta fA\}$. But the strictness of g, h(y) is a ξ-cluster since $y = \cap \{Cl_\eta fA : A \in h(y)\}$. Consequently $h:Y \to X^*$ is bijective and $j = h \circ f$. To show that $h:(Y,\eta) \to (X^*,(\xi^*)_t)$ is a homeomorphism, let $A \subset X$ and $y \in Y$. Then $y \in Cl_\eta fA$ iff $A \in h(y)$ iff $h(y) \in Cl_{\xi^*} jA$. Since $f:(X,\zeta) \to (Y,\eta)$ and $j:(X,\zeta) \to (X^*,(\xi^*)_t)$ are strict extensions, $h:(Y,\eta) \to (X^*,(\xi^*)_t)$ is a homeomorphism.

(c) The first part of (1) follows immediately 3.10 (2) and 4.6. The second part follows the fact that the N-space (X,ξ) constructed in (b) is contigual, provided that (Y,η) is compact.

(d) (2) is similar to (1).

Finally, an N-space is called a N3-space iff it is a regular N1-space.

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國 文 抄 錄

本 論文에서는, γ-cocluster 를 利用하여 N-空間의 completion 을 구상하고, 또한 主된 결과들은,

1) Concrete N1 -구조가 strict extension 을

2) Contigual N1 -구조가 strict compactification 을

3) N3-구조가 regular extension 을 각각 조사하는데 적절한 도구가 됨을 보였다.