On Ideals in a Polynomial Halfring

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多項式 準環의 Ideal에 관하여

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I. Introduction

H. E. Stone defined a *type of a k–ideal* K in a halfring H to be the set of all k–ideals I in the ring of differences $\bar{H}$ such that $I \cap H = K$. While the concept of ideal type is applicable to ideals in any halfring, it is of interest to consider ideal types in polynomial halfrings. The purpose of this paper is to consider a special ideal type found in polynomial halfrings. And we have some properties on relations between ideals in a halfring H and ideals in a polynomial halfring $H[x_1, x_2, \cdots, x_n]$.

A halfring is a triple (H, +, ·), where (H, +) is a commutative cancellative semigroup with identity (called zero) and (H, ·) is a semigroup whose multiplication destributes over the addition from both sides. A halfring H generates a ring of differences $\bar{H}$ unique up to isomorphism. A halfring H is said to be *strict* if a, $b \in H$ and $a+b=0$ implies $a=b=0$. The operations of a halfring H are extended to subsets in the familiar way. Then a *subsemiring* of H is a subset S with $0 \in S$, with $S+S \subset S$ and with $SS \subseteq S$. If also $SH \subset S$ or $HS \subseteq S$ or both, then S is a *left ideal* or *right ideal* or *ideal* respectively. An ideal I in a halfring H will be called a *k–ideal* if $a \in I$, $b \in H$ and $a+b \in I$ imply $b \in I$. Let A be an ideal in H. Then the ideal $\bar{A}_k = \cap \{B : B$ is a k–ideal and $A \subset B\}$ will be called k–*closure* of A.

Definition 1.1 (H. Stone) Let H be a halfring and A an ideal in H. The *ideal type* of A, denoted by $\tau(A)$, is the set of all ideals I in the ring of differences $\bar{H}$ such that $I \cap H = \bar{A}_k$.

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Ⅱ. Ideal Structure in a Polynomial Halfring

Throughout this paper, unless otherwise stated, H will be a commutative halfring with an identity.

Let $x_1, x_2, \cdots, x_n$ be indeterminates which commute with each other and with each element of H. It is clear that $H[x_1, x_2, \cdots, x_n]$ is a halfring, and that a typical element in $H[x_1, x_2, \cdots, x_n]$ is of the form

$$a_{i_1 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

with $a_{i_1 i_2 \cdots i_n} \in H$

The following notation will be used throughout hereafter. Let S(r, n) denote the set of sequence of length n from the integers 0, 1, 2,⋯,r, and $\Phi = x_1 x_2 \cdots x_n$. For $\alpha = \{i_1, i_2, \cdots, i_n\} \in S(r, n)$, let $a_\alpha = a_{i_1 i_2 \cdots i_n}$ and $\Phi^\alpha \triangleq x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$.

With this notation, the polynomial

$$\Sigma a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_n}$$

can be written $\Sigma a_\alpha \Phi^\alpha$, $\alpha \in S(r, n)$. Also, let $S(\infty, n)$ denote the set of sequence of length n from the set of nonnegative integers. It is clear that $S(r, n) \subset S(\infty, n)$.

A collection of sets $G = \{A_\alpha : \alpha \in S(\infty, n)$ and n is fixed$\}$ will be called a *generalized ascending chain* if A_α and A_β are two sets in G such that $\alpha = \{i_1, i_2, \cdots, i_n\}$, $\beta = \{j_1, j_2, \cdots, j_n\}$ and $i_k \leqq j_k$ for each k=1, 2,⋯,n, then $A_\alpha \subseteq A_\beta$. In the followings, we denote $\alpha \leqq \beta$ if $i_k \leqq j_k$ for all k=1, 2,⋯,n.

An ideal M in $H[x_1, x_2, \cdots, x_n]$ will be called *monic* if

$f(x_1, x_2, \cdots, x_n) = \Sigma a_\alpha \Phi^\alpha \in M$

implies that each $a_\alpha \Phi_\alpha \in M$. An ideal A in a halfring $H[x_1, x_2, \cdots, x_n]$ will be called a *weak k—ideal* if there exists $\alpha \in S(\infty, n)$ such that A is a k—ideal with respect to all $f \in A$ with degree $f \leqq \alpha$.

Definition 2. 1. Let A be an ideal in $H[x_1, x_2, \cdots, x_n]$. The *ideal type* of A in $H[x_1, x_2, \cdots, x_n]$, denoted by $\tau_0(A)$, is the set of all weak k-ideals I in $H[x_1, x_2, \cdots, x_n]$ such that $A \subset I \subset \bar{A}_k$. Then the ideal type $\tau_0(A)$ of A in $H[x_1, x_2, \cdots, x_n]$ is closed under unions of chains and arbitrary intersections, and contains $\bar{A}_k$ as a maximal elements.

Let A be an ideal in $H[x_1, x_2, \cdots, x_n]$. For $\alpha \in S(\infty, n)$ is the sequence of fixed integers, and

$A_\alpha = \{f : f \in A$ and degree of $f \leqq \alpha\}$:

the ideal

$\bar{A}_{k\alpha} = \cap \{B_i :$ B_i is a weak k—ideal with k—degree at least α and $A_\alpha \subset B_i\}$

will be called the *weak k—closure* of A. It is clear that $\bar{A}_{k\alpha}$ has k-degree at least α and $A_\alpha \subset \bar{A}_{k\alpha} \subset \bar{A}_k$.

Theorem 2.2. Let $A = \{A_\alpha : \alpha \in S(\infty, n)\}$ be a generalized ascending chain of k—ideals in a halfring H. Then

$M(A) = \{\Sigma a_\alpha \Phi_\alpha \in H[x_1, x_2, \cdots, x_n] : a_\alpha \in A_\alpha\}$

is a monic k-ideal in $H[x_1, x_2, \cdots, x_n]$.

Proof. Let $f = \Sigma a_\alpha \Phi^\alpha$ and $g = \Sigma b_\alpha \Phi^\alpha$ be elements of M(A) and $h = \Sigma c_\beta \Phi^\beta \in H[x_1, x_2, \cdots, x_n]$. Also let $\alpha = \{i_1, i_2, \cdots, i_n\}$ and $\beta = \{j_1, j_2, \cdots, j_n\}$. Since A_α is a k—ideal in H, it follows that $a_\alpha + b_\alpha \in A_\alpha$ and consequently $f+g \in M(A)$. Now $hf = \Sigma d_w \Phi^w$ where $w = \{r_1, r_2, \cdots, r_n\}$, and $d_w = \Sigma c_\beta a_\alpha$ for $w = \alpha + \beta$. Clearly, $r_m = i_m + j_m$ and $i_m \leqq r_m$ for each m, and it follows that $a_\alpha \in A_w$ since A is a generalized ascending chain of ideals. Thus $d_w \in A_w$ and $hf \in M(A)$. Therefore M(A) is a monic ideal. Suppose that each A_α is a k-ideal. $f+g \in M(A)$ and $f \in M(A)$. Then $a_\alpha + b_\alpha \in A_\alpha$, $a_\alpha \in A_\alpha$ and it follows that $b_\alpha \in A_\alpha$. Consequently, $b_\alpha \Phi^\alpha \in M(A)$ and $g = \Sigma b_\alpha \Phi^\alpha \in M(A)$. Therefore M(A) is a k—ideal.

Proposition 2.3. Let A be an ideal in $H[x_1, x_2,$

$\cdots,x_n]$. For each $\alpha \in S(\infty, n)$, let

$C(A)_\alpha = \{a \in H$: there is an $f \in A$ such that $a\Phi^\alpha$ is a term of $f\}$

and $G = \{C(A)_\alpha\}$. Then we have that G is a generalized ascending chain of ideals in H.

Proof. For $a \in C(A)_\alpha$ and $b \in C(A)_\alpha$ there are polynomials f and g in A such that $a\Phi^\alpha$ and $b\Phi^\alpha$ are terms of f and g respectively. Now $f+g \in A$ and $(a+b)\Phi^\alpha$ is a term of $f+g$ and it follows that $a+b \in C(A)_\alpha$. If $c \in H$ then $cf \in A$ and $ca\Phi^\alpha$ is a term of cf. Consequently, $ca \in C(A)_\alpha$. Hence each $C(A)_\alpha$ is an ideal in H. Next, suppose that $C(A)_\alpha$ and $C(A)_\beta$ are such that $\alpha = \{i_1, i_2, \cdots, i_n\}$, $\beta = \{j_1, j_2, \cdots, j_n\}$, $i_k \leqq j_k$ for each k and $b \in C(A)_\alpha$. Then there is a polynomial $f \in A$ such that $b\Phi^\alpha$ is a term of f. Let $\beta - \alpha = \{j_1-i_1, j_2-i_2, \cdots, j_n-i_n\}$. Then $\Phi^{\beta-\alpha} f \in A$ with $\Phi^{\beta-\alpha} b\Phi^\alpha = b\Phi^\beta$ as one of its terms. Thus $b \in C(A)_\beta$ and it follows that $C(A)_\alpha \subseteq C(A)_\beta$ and G is a generalized ascending chain of ideals in H.

We call these ideals $C(A)_\alpha$ *the coefficient ideals* of A.

Proposition 2.4. Let A be an ideal in $H[x_1, x_2, \cdots,x_n]$. Then $\{\bar{A}_{k\alpha}\}$ is a generalized ascending chain of weak k-ideals such that $\bar{A}_{k\alpha} \subset \bar{A}_k$ for each $\alpha \in S(\infty, n)$.

Proof. By definition, $\bar{A}_{k\alpha}$ is a weak k-ideal in $H[x_1, x_2, \cdots,x_n]$ for each $\alpha \in S(\infty, n)$. Suppose that A_α and A_β are such that $\alpha = \{i_1, i_2, \cdots, i_n\}$, $\beta = \{j_1, j_2, \cdots, j_n\}$, $i_k \leq j_k$ for each k, and $f \in \bar{A}_{k\alpha}$. Then $f \in B_i$ for all B_i, which is a weak k-ideal with k-degree at least α and $A_\alpha \subset B_i$. If C_j is a weak k-ideal with k-degree at least β and $A_\beta \subseteq C_j$, then C_j is one of the $\{B_i\}$. Hence $f \in C_j$ for all j, and

$\bar{A}_{k\beta} = \cap \{C_j$: C_j is a weak k-ideal with degree at least β and $A_\beta \subset C_j\}$

contains f. Therefore $\bar{A}_{k\alpha} \subset \bar{A}_{k\beta}$, and $\{\bar{A}_{k\alpha}\}$ is a generalized ascending chain of weak k-ideals.

Proposition 2.5. Let H be a strict halfring and let A be an ideal in $H[x_1, x_2, \cdots,x_n]$. If $W_\alpha = A + \bar{A}_{k\alpha}$, then $\{W_\alpha\}$ is a generalized ascending chain of ideals in $\tau_0(A)$ such that the k-degree of W_α is at least α. And if $I \in \tau_0(A)$, there exists $\gamma \in S(\infty, n)$ such that $W_\gamma \subset I$.

Proof. Since $A_\alpha \subset \bar{A}_{k\alpha}$ is a weak k-ideal with k-degree at least α, if $f \in A_\alpha$, then $f \in W_\alpha$, and degree $f \leqq \alpha$ imply $f \in \bar{A}_{k\alpha}$. If $f \in W_\alpha$ and degree f $\leqq \alpha$, then $f \in \bar{A}_{k\alpha}$ since H is strict. Hence W_α is a k-ideal with respect to all polynomials of degree less than or equal to α and it follows that W_α is a weak k-ideal with k-degree at least α. By proposition 2.4, $\{W_\alpha\}$ is a generalized ascending chain of ideals in $\tau_0(A)$. Now suppose $I \in \tau_0(A)$. If the k-degree of I is ∞, then $I = \bar{A}_k$ and $W_\alpha \subset I$ for all $\alpha \in S(\infty, n)$. If I has finite k-degree, say γ, then it is clear that $\bar{A}_{k\gamma} \subset I$ and consequently, $W_\gamma = A + \bar{A}_{k\gamma} \subset I$.

This proposition 2.5 implies that with each nonzero ideal A in $H[x_1, x_2, \cdots,x_n]$ there can be associated a unique number $\omega = | \{W_\alpha\} |$.

Definition 2.6. An ideal A in $H[x_1, x_2, \cdots,x_n]$ will be called a *quasi-k-ideal* if ω is finite.

Since $W_\alpha = A + \bar{A}_{k\alpha}$, it is clear that the number ω associated with A is the number of distinct weak k-closures of A.

Definition 2.7. Let A be an ideal in a halfring H. The set

$A' = \{x \in H$: there is an $a \in A$ such that $a + x \in A\}$

is called the ***k*-boundary** of A.

Lemma 2.8. If A is an ideal in a halfring H, then A' is a k-ideal in H and $A \subset A'$.

Proof. Given x_1 and $x_2 \in A'$, there are elements $a_1, a_2 \in A$ such that $a_1 + x_1 \in A$ and $a_2 + x_2 \in A$. Since $a_1 + a_2 \in A$ and

$(a_1+x_1)+(a_2+x_2)=(a_1+a_2)+(x_1+x_2) \in A$,

it follows that $(x_1+x_2) \in A'$. If $b \in H$, then

$b(a_1+x_1) = ba_1 + bx_1 \in A$.

Hence $bx_1 \in A'$ and A' is an ideal. Now suppose that $x \in A'$, $y \in H$ and $x+y \in A'$. Then there exist u and $v \in A$ such that $x+u \in A$ and $(x+y)+v \in A$. Now $(x+u)+v \in A$ and $[(x+y)+v]+u \in A$. But $[(x+y)+v]+u=[(x+u)+v]+y$. Consequently, $y \in A'$ and A' is a k-ideal. If $e \in A$, then $0+e \in A$ and it follows that $e \in A'$ and $A \subset A'$.

Corollary 2.9. An ideal A in H is a k-ideal if and only if $A=A'$.

Corollary 2.10. If A is an ideal in H, then $A'=\bar{A}_k$.

Lemma 2.11. If A and B are monic ideals in $H[x_1, x_2, \cdots, x_n]$, then $A=B$ if and only if $C(A)_\alpha=C(B)_\alpha$ for each $\alpha \in S(\infty, n)$.

Proof. If $A=B$ and a polynomial $f \in A$, then any coefficient a_α of f is in both $C(A)_\alpha$ and $C(B)_\alpha$. Hence $C(A)_\alpha=C(B)_\alpha$ for $\alpha \in S(\infty, n)$. Conversely, suppose that $C(A)_\alpha=C(B)_\alpha$ for $\alpha \in S(\infty, n)$ and $f=\Sigma a_\alpha \Phi^\alpha \in A$. Then $a_\alpha \in C(B)_\alpha$ for each α, and there are polynomials $f_\alpha \in B$ such that $a_\alpha \Phi^\alpha$ is a term of f_α. Since B is a monic ideal, it follows that each $a_\alpha \Phi^\alpha \in B$. Hence $f=\Sigma a_\alpha \Phi^\alpha \in B$ and $A \subset B$. Similarly $B \subset A$, and it follows that $A=B$.

Theorem 2.12. If a halfring H is Noetherian, then every ascending chain of monic ideals in $H[x_1, x_2, \cdots, x_n]$ is finite.

Proof. Let H be Noetherian, and $\{A_\alpha\}$ be a generalized ascending chain of monic ideals in $H[x_1, x_2, \cdots, x_n]$. Consider the corresponding coefficient ideals $\{C(A_\alpha)_\beta\}$ in H. Then these ideals form a double array of generalized ascending ideals, that is, if $\alpha=\{i_1, i_2, \cdots, i_n\}$, $\beta=\{j_1, j_2, \cdots, j_n\}$, $\gamma=\{k_1, k_2, \cdots, k_n\}$ and $\delta=\{l_1, l_2, \cdots, l_n\}$ such that $i_m \leqq k_m$ and $j_m \leqq l_m$ for $m=1, 2, \cdots, n$, then $C(A_\alpha)_\beta \subset C(A_\gamma)_\delta$. Since H is Noetherian, there exists $\mu \in S(\infty, n)$ such that for each $\nu > \mu$, and all $\beta \in S(\infty, n)$, $C(A_\mu)_\beta=C(A_\nu)_\beta$. Hence Lemma 2.11 assures that $A_\nu=A_\mu$ for each $\nu \leqq \mu$.

Lemma 2.13. Let H be a halfring and let A be a monic ideal in $H[x_1, x_2, \cdots, x_n]$. Then $\bar{A}_k$ is a monic ideal.

Proof. Consider the k-boundary A' of A;

$A'=\{g \in H[x_1, x_2, \cdots, x_n]$: there is $f \in A$ such that $f+g \in A\}$.

By corollary 2.10, $\bar{A}_k=A'$. Let $g=\Sigma b_\alpha \Phi^\alpha \in \bar{A}_k$. Then there exists a polynomial $f=\Sigma a_\alpha \Phi^\alpha \in A$ such that

$f+g=\Sigma(a_\alpha+b_\alpha)\Phi^\alpha \in A$.

It could happen that $f=0$. Since A is monic, $(a_\alpha+b_\alpha)\Phi^\alpha \in A$ for each α. Now $a_\alpha \Phi_\alpha \in A$ and it follows that $b_\alpha \Phi^\alpha \in A \subset \bar{A}_k$, and $\bar{A}_k$ is a monic ideal.

Theorem 2.14. If H is a Noetherian halfring, then every monic ideal in $H[x_1, x_2, \cdots, x_n]$ is a quasi-k-ideal.

Proof. Let A be a monic ideal and $W_\alpha \in \tau_o(A)$. Then $\bar{A}_k$ is a monic ideal by lemma 2.13. Now let

$M_\alpha=\{f \in \bar{A}_{k\alpha}$: degree $f \leqq \alpha$, $\alpha \in S(\infty, n)\}$.

Then $\bar{A}_{k\alpha}$ is a monic ideal with respect to polynomials in M_α by a proof similar to that of lemma 2.13. If $G(M_\alpha)$ is the k-ideal generated by M_α, then $G(M)$ is a monic ideal,

$A_\alpha \subset M \subset G(M) \subset \bar{A}_{k\alpha}$, and the k-degree of $G(M)$ is at least α. Consequently, $G(M)=\bar{A}_{k\alpha}$, and $\bar{A}_{k\alpha}$ is a monic ideal. Hence $W_\alpha=A+\bar{A}_{k\alpha}$ is a monic ideal. Therefore $\{W_\alpha\}$ is a generalized ascending chain of monic ideals in $H[x_1, x_2, \cdots, x_n]$ and must necessarily be finite by theorem 2.12. Consequently, $\omega=|\{W_\alpha\}|$ is finite and A is a quasi-k-ideal.

Corollary 2.15. If $H[x_1, x_2, \cdots, x_n]$ is a Noetherian halfring, then every ideal in $H[x_1, x_2, \cdots, x_n]$ is a quasi-k-ideal.

Example 2.16. Consider the ideal $A=(5, x^2y^3+5)$ in $Z^+[x, y]$. Now A is neither a monic ideal nor a k-ideal since $x^2y^3 \notin A$. Any polynomial of degree $\beta=(i, j)$ such that $i<2$, $j<3$ in A is of the form.

$5c_{00}+5c_{10}x+5c_{01}y+5c_{11}xy+5c_{02}y^2+5c_{12}xy^2$

for $c_{ij} \in Z^+$ and if follows that A is a k-ideal with respect to these polynomials since each $5c_{ij}x^iy^j \in A$. Consequently, A is a weak k-ideal of degree $\alpha=(1, 2)$. From this it follows that

$$\bar{A}_{k(0,0)}=\bar{A}_{k(0,1)}=\bar{A}_{k(0,2)}=\bar{A}_{k(1,0)}=\bar{A}_{k(1,1)}=\bar{A}_{k(1,2)}\subsetneq A$$

and hence $W_{(1,2)}=W_\beta=A$ for all $\beta=(i, j)<(2, 3)$. Now any k-ideal containing 5 and x^2y^3+5 must contain x^2y^3. Since $(5, x^2y^3)$ is a k-ideal, it is clear that $\bar{A}_{k\alpha}=(5, x^2y^3)$, where $\alpha=(2, 3)$. This gives $W_\alpha=\bar{A}_{k\alpha}$ and consequently, $W_\gamma=W_\alpha$ for all $\gamma=(i, j)$ such that $i\geqq 2$, $j\geqq 3$. Therefore $||W_\alpha||=\omega=2$ and A is a quasi-k-ideal.

Ⅲ. On Relation between Ideals in H and Ideals in $H[x_1, x_2,\cdots,x_n]$

Let I be an ideal in $\bar{H}$, E be an ideal in H. Then $I\cap H=E$ if and only if $I[x_1, x_2,\cdots\cdots x_n]\cap H[x_1, x_2,\cdots,x_n]=E[x_1, x_2,\cdots,x_n]$. Hence $I\in\tau(E)$ if and only if $I[x_1, x_2,\cdots,x_n]\in\tau(E[x_1, x_2,\cdots,x_n])$, and it follows that the map $I \rightarrow I[x_1, x_2,\cdots,x_n]$ is lattice isomorphism between $\tau(E)$ and a subset $\tau(E[x_1, x_2,\cdots,x_n])$.

Theorem 3.1. Let H be a halfring with an identity, I(H) the collection of all ideals in H and $I(H[x_1, x_2,\cdots,x_n])$ the collection of all ideals in $H[x_1, x_2,\cdots,x_n]$. The map

$f: I(H[x_1, x_2,\cdots,x_n])\rightarrow I(H)$ given by $f(A)=C(A)$ induces an equivalence relation on $I(H[x_1, x_2,\cdots,x_n])$, where $C(A)=\bigcup C(A)_\alpha$. Moreover, there is a one to one correspondence between I(H) and the set of equivalence classes of $I(H[x_1, x_2,\cdots,x_n])$.

Proof. Since each ideal A in $H[x_1, x_2,\cdots,x_n]$ can be associated with a unique ideal in H (by proposition 2.3), namely C(A), f is well-defined and f is surjective. Define a relation R in $I(H[x_1, x_2,\cdots,x_n])$ as follows;

$(A, B)\in R$ if and only if $C(A)=C(B)$ if and only if $A, B\in f^{-1}(C(D))$ for some $C(D)\in I(H)$.

Then R is an equivalence relation and consequently partitions $I(H[x_1, x_2,\cdots,x_n])$ into equivalence classes. If P is an equivalence class of $I(H[x_1, x_2,\cdots,x_n])$ and $A\in P$ then $P=f^{-1}(C(A))$. Thus the equivalence classes are induced by f. Let $I(H[x_1, x_2,\cdots,x_n])/R$ be the set of equivalence classes of $I(H[x_1, x_2,\cdots,x_n])$, and let

$f^*: I(H)\rightarrow I(H[x_1, x_2,\cdots,x_n])/R$

be given by, $f^*(C(A))=P$,

where P is the equivalence class containing A. Then f^* is bijective. For, if $P\in I(H[x_1, x_2,\cdots,x_n])/R$ and $A\in P$, then $P=f^{-1}(C(A))$ and it follows that $f^*(C(A))=P$. If $f^*(C(A))=f^*(C(B))$, then

$P=f^{-1}(C(A))=f^{-1}(C(B))=Q$,

where Q is the equivalence class containing B. Therefore $(A, B)\in R$, and hence $C(A)=C(B)$.

Corollary 3.2. Let H be a halfring with an identity and $\bar{H}$ its ring of differences. If $\bar{I}(H)$ is the collection of all ideals in $\bar{H}$ and $\bar{I}(H[x_1, x_2,\cdots,x_n])$ the collection of all ideals in $\bar{H}[x_1, x_2,\cdots,x_n]$, then the map $\bar{f}: \bar{I}(H[x_1, x_2,\cdots,x_n])\rightarrow\bar{I}(H)$ given by $\bar{f}(A)=C(A)$ induces an equivalence relation on $\bar{I}(H[x_1, x_2,\cdots,x_n])$. Moreover, there is a one to one correspondence between $\bar{I}(H)$ and the set of equivalence classes of $\bar{I}(H[x_1, x_2,\cdots,x_n])$.

Proposition 3.3. If A is an ideal in $H[x_1, x_2,\cdots,x_n]$, then $\bar{f}(\bar{A})=\overline{f(A)}$.

Proof. Since $\bar{f}(\bar{A})=C(\bar{A})$ and $\overline{f(A)}=\overline{C(A)}$, it suffices to show that $\overline{C(A)}=C(\bar{A})$. Suppose $d\in C(\bar{A})$. Then $d\in C(\bar{A})$ for some $\beta\in S(\infty, n)$ and it follows that $d\Phi^\beta$ is a term of f-g, where $f=\Sigma a_\alpha\Phi^\alpha$, $g=\Sigma b_\alpha\Phi^\alpha\in A$. Thus $d\Phi^\beta=(a_\beta-b_\beta)\Phi^\beta$ and $d=a_\beta-b_\beta\in\overline{C(A)}_\beta\subset\overline{C(A)}$. Conversely, if $d\in\overline{C(A)}$, then $d=a-b$, where $a, b\in C(A)=\bigcup C(A)_\alpha$. Thus $a\in C(A)_\beta$, $b\in C(A)_\gamma$ for some β, $\gamma\in S(\infty, n)$. Since $\{C(A)_\alpha\}$ is a generalized ascending chain, either $C(A)_\beta\subset C(A)_\gamma$ or $C(A)_\gamma\subset C(A)_\beta$, say $A_\gamma\subset A_\beta$. There exist polynomials $f, g\in A$ such that $a\Phi^\beta$ and $b\Phi^\beta$ are terms of f and g, respectively. Consequently, $d\Phi^\beta=a\Phi^\beta-b\Phi^\beta$ is a term of $f-g\in\bar{A}$ and it follows that $d\in C(\bar{A})_\beta$

$\subset C(\bar{A})$. Therefore $\overline{C(A)} = C(\bar{A})$.

Theorem 3.4. Let H be a Noetherian halfring and $f^{-1}(C(A)) = P \in I(H[x_1, x_2, \cdots, x_n])/R$ for an ideal A in $H[x_1, x_2, \cdots, x_n]$. Then $B \in P$ if and only if there exists $\beta \in S(\infty, n)$ such that $C(B)_\gamma = C(A)_\beta$ for all $\gamma \in S(\infty, n)$, where $\beta \leqq \gamma$.

Proof. Since $\{C(A)_\alpha\}$ is a generalized ascending chain in H, it follows that there exists $\beta = \{i_1, i_2, \cdots, i_n\}$ such that $C(A)_\gamma = C(A)_\beta$ for all $\gamma = \{j_1, j_2, \cdots, j_n\}$ such that $j_k \geqq i_k$. Consequently, all the coefficients of terms of degree γ of a polynomial $f \in A$ lie in a fixed ideal $C(A)_\gamma$. This condition characterizes the class $P = f^{-1}(C(A))$, i. e. $B \in P$ if and only if there exists $\beta = \{i_1, i_2, \cdots, i_n\} \in S(\infty, n)$ such that $C(B)_\gamma = C(A)_\beta$ for all $\gamma = \{j_1, j_2, \cdots, j_n\}$, where $j_k \geqq i_k$.

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國 文 抄 錄

本 論文에서는 多項式 準環에서 나타나는 weak–k–ideal과 quasi–k–ideal에 대하여 研究 조사하여 다음의 結果를 얻었다.

첫째, 一般 準環의 ideal들의 上昇 chain에서 多項式 準環의 ideal을 구성할 수 있음과 그 반대로 가능함을 밝혔다.

둘째, 多項式 準環內에서 weak–k–ideal을 使用하여 quasi–k–ideal을 정의하였고, 이를 Noetherian 準環에 應用하여 모든 monic ideal이 quasi–k–ideal이 됨을 證明하고 例를 보였다.

셋째, 多項式 準環內에 同値關係를 定義하고, 여기에서 만들어진 同値類들과 계수 準環의 ideal들이 일대일 對應關係에 있음을 밝혔고, 또 이 結果를 difference 環으로 확장하였으며, Noetherian準環에 應用하여 同値類의 分類方法을 제시하였다.